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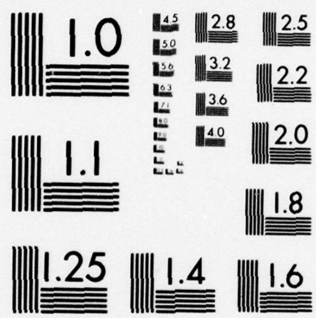


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# AN ANALYSIS OF ONE WAREHOUSE, N RETAILER PRODUCTION INVENTORY SYSTEMS

by

Howard Mark Singer

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# AN ANALYSIS OF ONE WAREHOUSE, N RETAILER PRODUCTION INVENTORY SYSTEMS

Howard Mark Singer, Ph.D.
Cornell University 1979

In this thesis we examine the one warehouse, N retailer production inventory system. We present properties of optimal operating policies and methods to determine various operating policies given a fixed cost for set-up and an inventory carrying charge at each facility. We assume that the external demands on this system occur either at a known continuous rate that is stationary over an infinite time horizon, or at a known rate that may vary in each of a finite number of periods. We will refer to the former case as a continuous one warehouse, N retailer problem and the latter as a dynamic demand problem.

A one warehouse, N retailer system is a special case of the more general arborescent production inventory system. We begin by examining the previous research on systems with this arborescent structure as well as reviewing the literature dealing with the serial and assembly multi-echelon production inventory systems.

The continuous one warehouse, N retailer problem is then examined in detail. The basic model is introduced as well as some previously solved special cases. Several basic production policies that have been suggested for this system are reviewed and properties of an overall optimal solution are discussed. We develop optimal and heuristic algorithms to determine the values of the parameters in "single cycle" policies (Schwarz, 1973)

and make comparisons based on both the quality of solutions obtained and the computational effort. It has been conjectured by Graves and Schwarz (1977) that these single cycle policies are optimal for certain larger classes of production plans and we demonstrate that this is not the case. Finally, we discuss the class of multiple cycling policies and their relation to optimality.

We then explore the dynamic one warehouse, N retailer problem. The basic model is introduced and both the Wagner-Whitin (1958) and the Zangwill (1966) algorithms are reviewed. Veinott (1969) has suggested the extension of Zangwill's algorithm to the case of N retailers, and we present this extension along with computational simplifications. Easily computed upper and lower bounds on the optimal solution value are developed. Exact solution techniques for several special cases for the cost structure are also discussed. Finally, areas for future investigation are suggested.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

# CHAPTER 1

# INTRODUCTION

## 1.1 General Introduction

In this thesis we examine the one warehouse, N retailer production inventory system. We present properties of optimal operating policies and methods to determine various operating policies given a fixed cost for set-up and an inventory carrying charge at each facility. We assume that the external demands on this system occur either at a known continuous rate that is stationary over an infinite time horizon, or at a known rate that may vary in each of a finite number of periods. We will refer to the former case as a continuous one warehouse, N retailer problem and the latter as a dynamic demand problem.

A one warehouse, N retailer system is a special case of the more general arborescent production inventory system. In the next section we examine the previous research on systems with this arborescent structure as well as review the literature dealing with the serial and assembly multi-echelon production inventory systems. Particular emphasis will be placed on the assumptions in these works that are common to the research in this thesis.

In Chapter 2 the continuous one warehouse, N retailer problem is examined in detail. The basic model is introduced as well as some previously solved special cases. Several basic production policies that have been suggested for this system are reviewed and properties of an overall optimal solution are discussed. We develop optimal and heuristic algorithms to determine the values of the parameters in "single cycle"

policies (Schwarz, 1973) and make comparisons based on both the quality of the solutions obtained and the computational effort. It has been conjectured by Graves and Schwarz (1977) that these single cycle policies are optimal for certain larger classes of production plans and we demonstrate that this is not the case. Finally, we discuss the class of multiple cycling policies and their relation to optimality.

In Chapter 3 we explore the dynamic one warehouse, N retailer problem. The basic model is introduced and both the Wagner-Whitin (1958) and the Zangwill (1969) algorithms are reviewed. Veinott (1969) has suggested the extension of Zangwill's algorithm to the case of N retailers, and we present this extension along with computational simplifications. Easily computed upper and lower bounds on the optimal solution value are developed. Finally, exact solution techniques for several special cases for the cost structure are discussed.

In Chapter 4 we summarize the major results. In addition, we suggest areas for future investigation.

## 1.2 Serial, Assembly and Arborescent Production Inventory Systems

A multi-echelon production inventory system is one in which products are partially processed at certain facilities. Each subsequent facility performs additional processing until the product, or products, is available to satisfy external demands. Work in process inventories are permitted between the facilities. The problem is to determine an operating policy that meets the external demands while minimizing production and inventory costs.

All the research on multi-echelon production inventory systems is based, at least in part, on the methods used in explicitly solving the single facility problem. Single facility problems have been studied extensively and excellent summaries of the results on these systems can be found in the books by Hadley and Whitin (1963), and Johnson and Montgomery (1974). For the case of dynamic demands, the classic paper is by Wagner and Whitin (1958). Their results have been generalized many times to include multiple set-up costs (Lippman, 1969), piecewise concave costs (Love, 1973) and backlogging (Zangwill, 1966). One generalization of the basic single facility problem that has been extensively researched is the capacity constrained problem (Florian and Klein, 1971; Lambrecht and Vander Eecken, 1978; Baker, Dixon, Magazine and Silver, 1978). The Wagner-Whitin single facility algorithm and a capacity constrained single facility algorithm can be used to determine optimal policies in one warehouse, N retailer systems under certain cost restrictions. (These results will be developed in Chapter 3.)

One can envision many configurations for the facilities in a multi-echelon system; but, three particular structures have been extensively studied in the past. These three structures are the serial, assembly and arborescent systems (Figure 1.1). We will discuss each of these structures individually and summarize the relevant research.

In a serial multi-echelon system, each facility has only one immediate predecessor and one immediate successor. Work in process inventory flows from one facility to the next and at each facility additional processing is performed. External demands occur only at the final facility. A single conveyor belt assembly line with work stations along the route of the belt is an example of a serial multi-echelon system.

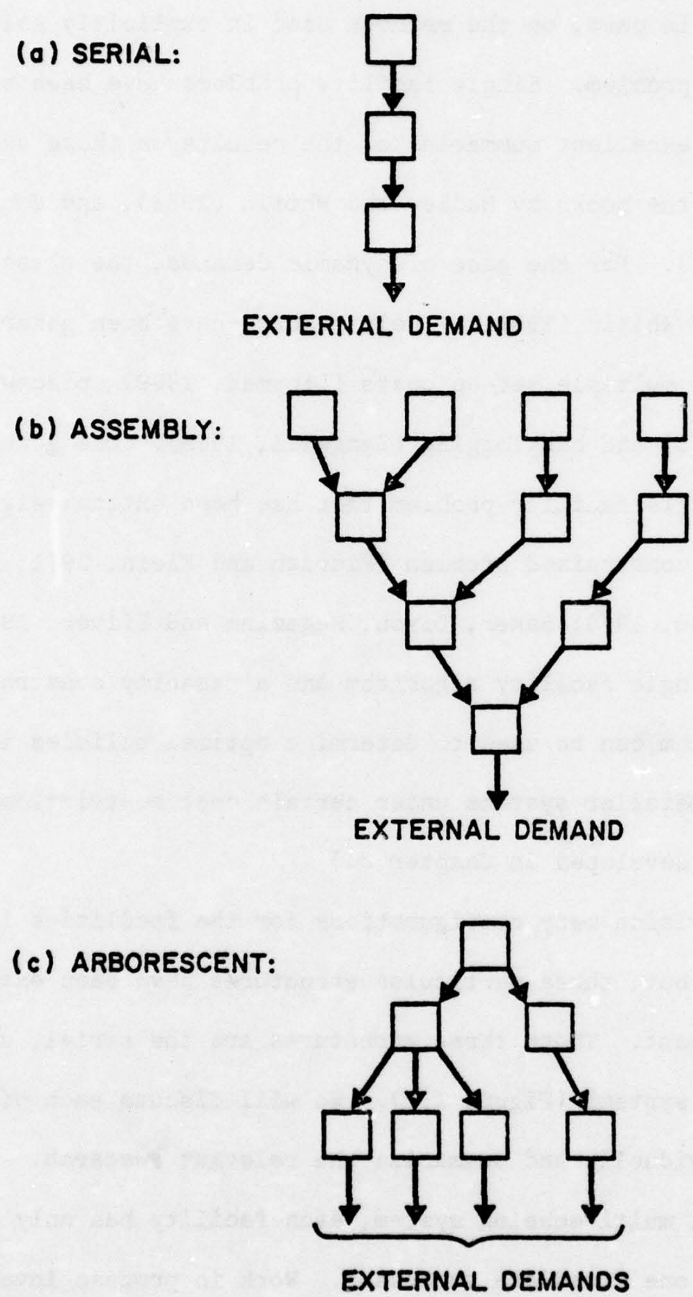



FIGURE 1.1 MULTI-ECHELON SYSTEM STRUCTURES

The research on serial production inventory systems with continuous deterministic demands includes two restrictive assumptions. Operating policies are chosen from among the policies in which there is a constant lot size at each stage and the lot size at a stage is an integer multiple of the lot size at the succeeding stage. Although this type of policy may be optimal for concave cost cases, there is no reason to assume it is optimal in a multistage system having constraints imposed by fixed production rates and lead times.

In addition to the assumptions discussed above, Taha and Skeith (1970) also assume that there is a delay between production and use of a lot and that backorders are permitted at the final stage. No explicit solution technique is given to determine the integers multiplying the lot sizes that specify the policy. Jensen and Khan (1972) examine a similar configuration but without the assumption of a delay between production and use of a lot. A discrete dynamic programming formulation to determine the optimal cycle length is developed for what is essentially a continuous parameter problem. Johnson (1972) provides expressions for the average in-process inventory levels with or without delays following production. A direct solution is obtained for the two stage system and an iterative procedure is developed for the three stage system.

For the case of dynamic demand, the classic reference is by Clark and Scarf (1960). They introduce the concept of "echelon stock" for computational simplicity and prove the optimality of a modified (s,S) policy for a serial system in which the demand in each period is determined by an observation from a known distribution. The echelon stock at facility i is inventory that has been processed by facility i and is

awaiting processing at any succeeding facility. That is, the facility $i$ echelon stock is inventory that has passed through facility $i$ but has not yet been used to satisfy external demands. An $(s,S)$ policy is being followed if, at review, the stock on hand plus on order, $i$, is less than $s$, then $S-i$ units are ordered; otherwise, no ordering is done. A more detailed discussion of $(s,S)$ inventory models can be found in the book by Tijms (1976).

For the serial system with deterministic dynamic demand, the basic references are by Zangwill (1966,1968,1969). He extends the single facility Wagner-Whitin model (1958) to the serial system with backlogging permitted at the final stage. Of particular interest are the representation of the problem as a minimum cost flow in a single source, concave cost network and the characterization of the solution by exact requirements, that is, the production quantities are always the sum of the requirements for a certain number of future consecutive periods. The network representation and exact requirements will be discussed in more detail in Chapter 3.

As in the single facility case, there have been many extensions to the basic serial system models. For example, Lambrecht and Vander Eecken (1977) have extended the Zangwill algorithm to include capacity constraints for facilities in series. Also, for the serial system with concave costs and storage costs non-decreasing in order of facilities and production costs non-increasing in time, Love (1972) has demonstrated that an optimal schedule has the property that if in a given period, a facility produces, then its successor does also. This nested structure is exploited in an algorithm for finding an optimal schedule.

In a multi-echelon assembly system each facility supplies at most one immediate successor, but it may be supplied by more than one immediate predecessor. External demands occur only at the final facility. Clearly, the assembly of components into subassemblies and eventually into one final product is an example of such a system. Also, the serial system is a special case of this more general configuration.

Crowston, Wagner and Williams (1973) investigate the continuous demand case with instantaneous production and integral lot sizes. Under the assumption of a stationary lot size at each facility, it is demonstrated that the optimal lot size at each facility is an integer multiple of the lot size at the successor facility. This fact is used in the construction of a dynamic programming algorithm for the computation of optimal lot sizes. Schwarz and Schrage (1975) omit the assumptions of instantaneous production and an integral lot size at the final stage and present a heuristic solution technique and a branch and bound algorithm to determine optimal lot sizes. In addition, Schwarz and Schrage assume that no lot splitting is permitted. That is, no part of a lot produced at a facility is available for use by its successor facility until the entire lot has been completed.

For the case of dynamic demand in a multi-stage assembly system, Crowston and Wagner (1973) present a dynamic programming algorithm and a branch and bound algorithm to determine optimal lot sizes that take advantage of Love type nested schedule for a particular cost structure.

In a multi-echelon arborescent system, each facility is supplied by at most one immediate predecessor, but it may, in turn, supply more than one immediate successor. External demands occur only at the facilities without successors. A national warehouse which supplies a group of

regional warehouses, each of which, in turn, supplies a group of retailers is an example of such a system. An arborescent system with only 2 echelons where the upper echelon consists of only a single facility is referred to as a one warehouse, N retailer system.

A limited amount of research has been conducted concerning one warehouse, N retailer and general arborescent systems. For the dynamic demand case, Veinott (1969) discusses the possible extension of Zangwill's algorithm to problems with an arborescent structure. Kalymon (1972) presents a decomposition algorithm which reduces the arborescent problem to a series of single stage problems at each lowest echelon facility. The dynamic demand case for the one warehouse, N retailer system will be investigated further in Chapter 3.

For the continuous demand case, Schwarz (1973) proposes the use of "single cycle" and "separate retailing" policies for the one warehouse, N retailer system. Attempts have been made (Graves and Schwarz, 1977) to relate these policies to more general arborescent structures and to determine the optimality of such solutions. A complete discussion of these policies will be presented in Chapter 2 along with a theoretical foundation detailing their relation to optimality.

# CHAPTER 2

## CONTINUOUS ONE WAREHOUSE, N RETAILER PROBLEMS

### 2.1 Introduction

In this chapter we examine the continuous one warehouse, $N$ retailer production inventory system. We begin by introducing the basic assumptions as well as some previously solved special cases. Schwarz (1973) has proposed two basic production policies called separate retailing and single cycle. We will reexamine both of these policies with particular emphasis on algorithms to determine the optimal single cycle policy. We will also carefully examine the optimality of single cycle policies for several wider classes of production plans.

### 2.2 Basic Model

The one warehouse, $N$ retailer system is a special case of the arborescent production inventory system discussed in Chapter 1. The lower echelon consists of $N$ "retailers", that is, those facilities that directly experience the external demand and the upper echelon consists of one "warehouse", which is the predecessor facility to all the retailers. Raw materials enter the warehouse, where some processing takes place. Units flow from the warehouse to the retailers where final processing takes place and these units then flow out of the system to satisfy external demand. We will refer to the warehouse as facility $0$ and we will refer to the $N$ retailers as facilities $1$ through $N$ respectively (Figure 2.1). This structure of facilities is common to both industrial and military situations.

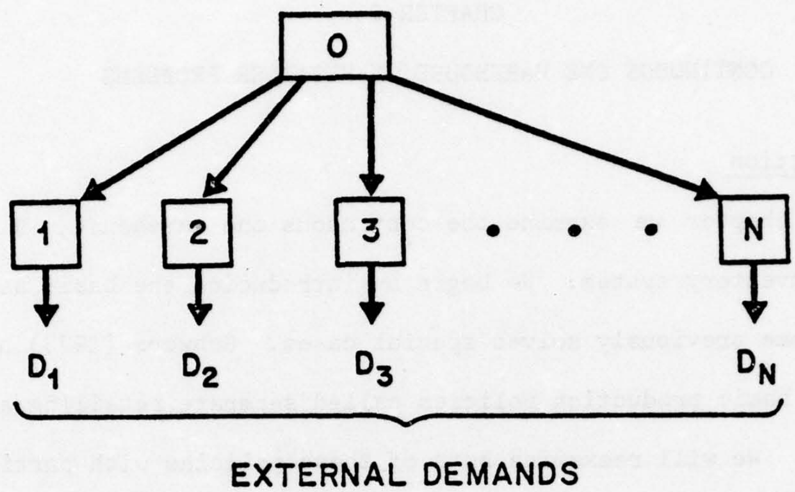



FIGURE 2.1 ONE WAREHOUSE, N RETAILER STRUCTURE

In the Air Force, for example, the lower echelon facilities are referred to as bases and the upper echelon as a depot (Muckstadt, 1977).

In the continuous demand case, the demand at retailer $i$ is assumed to occur at a known constant rate $D_i$. The demand rate may differ from one facility to another. We will assume that the warehouse experiences no external demand and thus the demand at the warehouse occurs at a constant rate, $D_0$, where

$$D_0 = \sum_{i=1}^{N} D_i . \tag{2.1}$$

At each facility there is a set-up cost and a holding cost. In particular, let

$K_i$ = Set-up cost incurred to produce a lot at facility $i$

and $h_i$ = Echelon holding cost per unit held at facility $i$,

where $i = 0,1,2,\ldots,N$.

It is important to point out that the echelon holding cost is charged against the echelon inventory, a concept first introduced by Clark and Scarf (1960). For the retailers, the echelon inventory is the on-hand inventory in the usual sense. For the warehouse, the echelon inventory is equal to the inventory held anywhere in the system. The use of echelon inventory in the model permits some very convenient simplifications but it has been pointed out that the use of echelon inventory instead of the more typical on-hand inventory may be confusing to anyone who wants to apply the model in practice (Szendrovits, 1978). Schwarz and Schrage (1978) have demonstrated that a policy which is optimal under the echelon stock charging scheme is also optimal under a charging scheme based upon the actual on-hand inventory.

With the cost structure defined in the above way, we can relax the restriction that only the retailers directly experience the external demand. If the external demand rate at the warehouse is $D_0'$, one can add to the problem an additional retailer with demand rate $D_0'$, a set-up cost of 0 and an echelon holding cost of 0. This creates a one-warehouse, N+1 retailer problem with no external demand at the warehouse which is equivalent to the original problem.

The assumption that no lot splitting is permitted is common to much of the research on multi-stage production inventory systems (Schwarz and Schrage, 1975). The no lot splitting assumption states that: (1) No part of a lot produced at a retailer may be used to satisfy external demand until the entire lot is produced and (2) No part of a

lot produced at the warehouse may be shipped to any retailer until the entire lot is produced. This assumption simplifies the computational effort and is quite realistic in many cases. For example, the warehouse and retailers may be physically far apart and it is logical that units will be shipped in bulk rather than individually. For the present, we will also assume that no lot splitting is permitted, although we will examine the alternative in a later section.

In addition, we will assume that the delivery times between the warehouse and the retailers and between the retailers and the external customers are constant. Given this assumption along with no lot splitting, we can also assume that production is instantaneous. The results presented can easily be extended to the case of finite production rates. Under no lot splitting, no part of a lot can be used until the entire lot has completed production and the flow of units through the system will remain unchanged if we assume that production is instantaneous. Finally, due to the zero delivery times and instantaneous production rates, units can be made available to satisfy external demand at time zero. Thus, we may also assume that the initial system inventory is zero.

## 2.3 Properties of an Optimal Solution

We will begin our investigation of production plans for the one warehouse, N retailer production inventory system by introducing some properties that optimal policies must possess (Graves and Schwarz, 1977):

P1. Each facility produces (orders) only when its on-hand inventory is zero.

P2. The warehouse produces only when at least one of its successors produces.

P3. For each retailer, all lot sizes produced in the time interval between successive production runs at the warehouse are equal.

Properties P1-P3, stated in a slightly more general fashion, will hold for the optimal policies of general arborescent systems as well. The proofs of properties P1-P3 may be found in Schwarz (1973).

We will define a simultaneous production point for a retailer $j$ as a point in time when stage $j$ and the warehouse produce simultaneously. There is, clearly, a simultaneous production point at time zero for all retailers in any feasible plans since we have assumed initial inventories of 0. In order to insúre the existence of a simultaneous production point, after time zero, it is necessary to assume that $h_0 > 0$. If $h_0 = 0$, the optimal policy for the warehouse is to produce an infinite lot size at time zero, regardless of the policy at any of the retailers. Thus, if $h_0 = 0$, the one warehouse, $N$ retailer problem becomes simply $N$ independent one-retailer problems which have been thoroughly analyzed (Hadley and Whitin, 1963). From this point on, we will assume that $h_0 > 0$. Given this assumption, we can add the following properties:

P4. The time interval between successive simultaneous production points is finite for each retailer.

P5. The lot sizes for each retailer are non-decreasing between its successive simultaneous production points.

Graves and Schwarz (1977) present proofs for properties P4 and P5. In the proof for P4, they incorrectly assume that the greatest lower bound on an infinite set of positive numbers is positive. A revised proof based in part on an incorrect proof by Graves and Schwarz (1978) is presented below.

<u>Proof of P4</u>. Consider some retailer violating P4 in an optimal policy P'. For simplicity, we will assume that the only simultaneous production point for retailer j is at time zero. (If this is not the case, a simple shift of the time axis will make the proof valid.) Let $I_j(t)$ be the echelon inventory of stage j at time t. Let $\{t_1,t_2,t_3,\ldots\}$ be the times when production takes place at the warehouse in P'. (The existence of such a countable set of production points is assured if the set-up cost at the warehouse or at all the retailers is strictly positive.) Let $I_j^{min} = \inf_i\{I_j(t_i)\}$. That is, $I_j(t_i)$ is the amount of inventory at retailer j at a point in time when there is production at the warehouse. There are two possible cases: Case 1, in which $I_j^{min} > 0$; and Case 2, in which $I_j^{min} = 0$. The proof for Case 1 is presented in Graves and Schwarz (1977).

The proof for Case 2 will construct a plan P with no greater cost than the assumed optimal policy P' and with a simultaneous production point after time 0. In P, we will reduce the amount produced at the retailer in P' for a particular point in time. This reduction will force a group of subsequent productions to be shifted earlier in time. This shift will create a simultaneous production point in P.

In particular, define $\overline{Q}_j = \sup_i\{Q_j(\tau_i')\}$ where $\tau_i'$ are the times when production takes place at j and $Q_j(\tau_i')$ is the lot size at retailer j at time $\tau_i$ in plan P'. Incidentally, $\overline{Q}_j < \infty$; otherwise, the cost of P' would be infinite, assuming of course the holding costs are positive. Let $\overline{T} = (h_j/h_0)\cdot\overline{Q}_j/D_j$. Intuitively, $\overline{Q}_j/D_j$ may be viewed as the maximum time interval between successive production runs at j.

Define $t_k$ to be the smallest $t_i \geq \overline{T}$. (Such a $t_k$ exists if the holding cost at the warehouse is non-zero.) Let $\varepsilon = \min_{i=1,2,\ldots,k}\{I_j(t_i)\} > 0$.

(If $\varepsilon = I_j(t_\ell) = 0$, then there is a simultaneous production at $t_\ell$).

Consider policy $P$ which is identical to $P'$ at the warehouse except that at time 0 the warehouse produces $\varepsilon$ less than in $P'$ and at time $t_k$ the warehouse produces $\varepsilon$ more. The echelon holding costs at the warehouse on the interval $(0,t_k)$ in $P$ are $\varepsilon \cdot t_k \cdot h_0$ less than the echelon holding costs on the same interval in $P'$. (See Figure 2.2.)

We will now show that it is possible to construct $P$ at retailer $j$ such that the increase in costs from $P'$ is no larger than the decrease in costs at the warehouse evaluated above. Let $\tau'_\ell$ be the earliest time after $t_k$ when production takes place at retailer $j$. Also let $\tau'_m = \min\{\tau'_i | Q_j(\tau'_i) > \varepsilon\}$. That is, $\tau'_m$ is the first production point for which retailer $j$ has lot size greater than $\varepsilon$. Since $I_j^{min} = \varepsilon$, $\tau'_m < t_2$ or $I_j(t_2) < \varepsilon$ contradicting the definition of $\varepsilon$. We will denote the production points at $j$, in the revised plan $P$, by $\tau_1, \tau_2, \ldots$ . Up until time $\tau'_m$, $P$ is identical to $P'$. That is, $\tau_i = \tau'_i$ and $Q_j^P(\tau_i) = Q_j(\tau'_i)$ for $1 \leq i < m$, where $Q_j^P(\tau_i)$ is the lot size, at retailer $j$, at time $\tau_i$ in the revised plan $P$. Let $\tau_m = \tau'_m$ but $Q_j^P(\tau_m) = Q_j(\tau'_m) - \varepsilon$. The conditions on $m$ will insure that this change maintains feasibility at $\tau'_m$. Since fewer units are produced at $\tau'_m$ to satisfy the demand at retailer $j$, the next production point, $\tau_{m+1}$ must occur earlier than $\tau'_{m+1}$ to maintain feasibility. Since $\varepsilon$ less units are produced at $\tau_m$, $\tau_{m+1} = \tau'_{m+1} - \varepsilon/D_j$ to insure feasibility. In fact, if we set $Q_j^P(\tau_i) = Q_j(\tau'_i)$ for $m < i < \ell$ then all production points $\tau_i$, for $m < i \leq \ell$, are shifted earlier in time than the corresponding production point in $P$ to maintain feasibility and they

are all shifted back by the same amount, $\varepsilon/D_j$. That is, $\tau_i = \tau_i' - \varepsilon/D_j$ for $m < i \leq \ell$. Finally, we add $\varepsilon$ units to the production at $\tau_\ell$. That is, $Q_j^P(\tau_\ell) = Q_j(\tau_\ell') + \varepsilon$. On the interval $(\tau_\ell',\infty)$, P is identical to P'.

To clarify this construction, at retailer j, consider Figure 2.2. In this particular case, it is assumed that $m = 4$, $k = 4$, $\ell = 7$ and $\varepsilon = I_j(t_2)$. The holding costs under both P' and P are illustrated, with the holding costs under the revised plan P denoted by the shaded area. P is identical to P' on $(0,\tau_4')$ and on $(\tau_7',\infty)$. At $\tau_4'$, $\varepsilon$ fewer units are produced in P and thus, retailer j reaches zero inventory at $\tau_5 = \tau_5' - \varepsilon/D_j$. $\tau_6$ and $\tau_7$ are also shifted $\varepsilon/D_j$ units back in time from $\tau_6'$ and $\tau_7'$. At $\tau_7$, $\varepsilon$ additional units are produced and thus at $\tau_7'$ plan P is again in phase with P'.

Let us denote $t_{min}$ as the production point in P' at the warehouse for which $I_j^{min} = I_j(t_{min}) = \varepsilon$. At $(t_{min} + \varepsilon/D_j)$, in P', retailer j has 0 inventory and thus, by the feasibility of P', $(t_{min} + \varepsilon/D_j) = \tau_i'$ for some $m < i \leq \ell$. Therefore, $t_{min} = \tau_i' - \varepsilon/D_j = \tau_i$ by the construction of P. Thus, $t_{min} = \tau_i$ is a simultaneous production point after time 0 in the plan P. In Figure 2.2, $I_j(t_2) = \varepsilon$ and $t_2 = \tau_5$ is a simultaneous production point in P.

It is obvious from Figure 2.2 that the holding cost in P is less than the holding cost in P' on the interval $(0,\tau_7)$ and equal to the costs in P' on $(\tau_7',\infty)$. In general, the only increase in costs, from P' to P, is on the interval $(\tau_\ell,\tau_\ell')$. In P, $Q_j(\tau_\ell')$ additional units are held over the interval $(\tau,\tau_\ell')$. Therefore the increase in costs can be written as

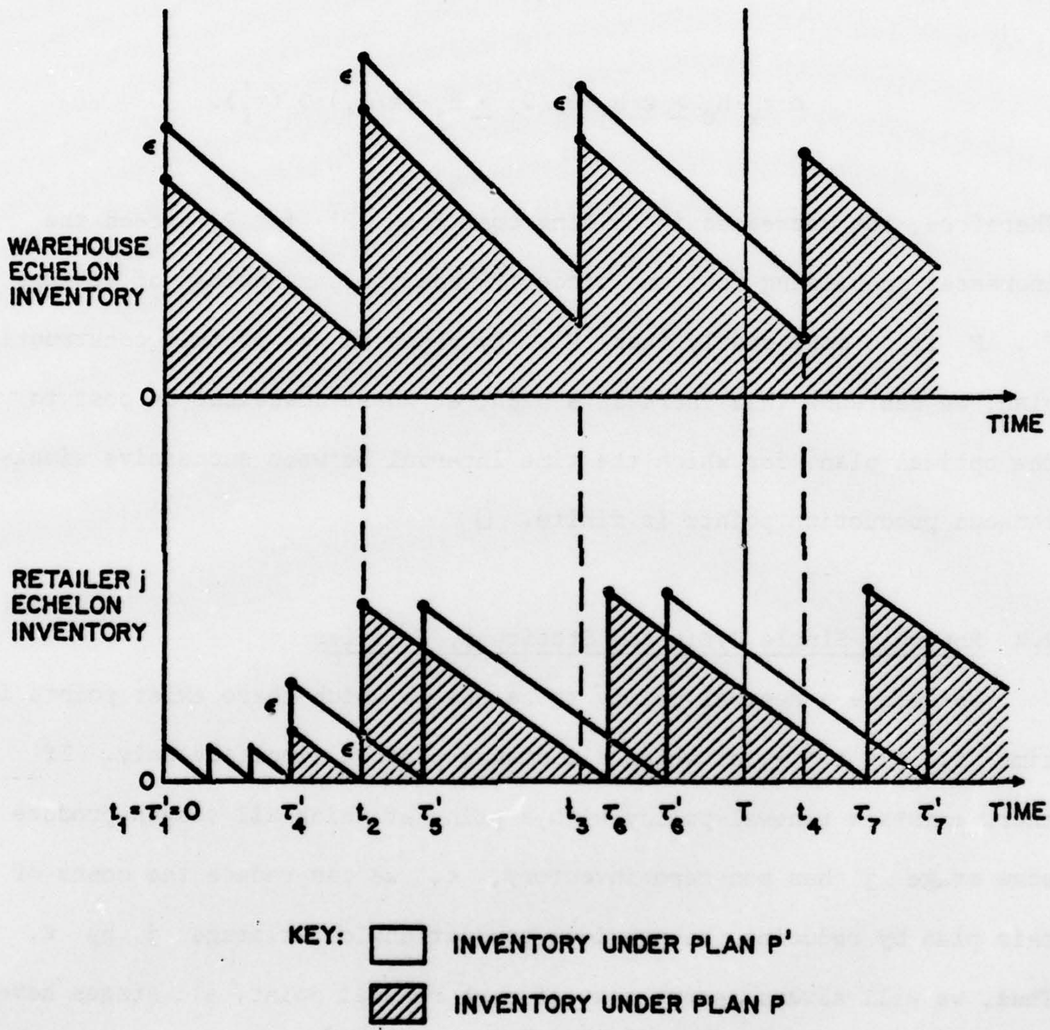



FIGURE 2.2 CONSTRUCTION OF REVISED POLICY FOR P4

$$h_j \cdot (\tau'_\ell - \tau_\ell) \cdot Q_j(\tau'_\ell) = h_j \cdot (\varepsilon/D_j) \cdot Q_j(\tau'_\ell).$$

Furthermore, by definition $Q_j(\tau'_\ell) \leq \overline{Q}_j$, and $t_k \geq \overline{T}$. Substituting in the definition of $\overline{T}$ and rearranging terms, we see

$$\varepsilon \cdot t_k \cdot h_0 \geq \varepsilon \cdot h_j \cdot \overline{Q}_j / D_j \geq h_j \cdot (\varepsilon/D_j) \cdot Q_j(\tau'_\ell).$$

Therefore, the decreases in holding cost from P' to P exceed the increases in holding cost and since P has the same number of set-ups as P', P is no more costly than P'. By repeated use of this construction plan, we can show that there is a plan, at worst identical in cost to the optimal plan, for which the time interval between successive simultaneous production points is finite. □

## 2.4 Renewal, Single Cycle and Stationary Policies

We define a renewal policy to be one in which there exist points in time $t$, $0 < t < \infty$, at which all stages produce simultaneously. If there exists a renewal policy with a point at which all stages produce and some stage $j$ has non-zero inventory, $\varepsilon$, we can reduce the costs of this plan by reducing the previous production lot at stage $j$ by $\varepsilon$. Thus, we will always assume that at each renewal point, all stages have zero inventory. Given this assumption, the system at a renewal point appears exactly as if we were starting the system again at time zero. If a renewal policy is the overall optimal policy and the first renewal occurs at $T$, we will clearly have renewals every $T$ time units and the policy will be the same between each pair of renewals. Furthermore,

in the best renewal policy the retailers produce the same lot size each time they produce. Alternatively, we can state,

Lemma 2.1. The optimal renewal policy is stationary at the retailers.

Proof of Lemma 2.1.

Suppose the optimal renewal policy is not stationary at retailer $j$, i.e. between a pair of renewals, retailer $j$ produces consecutive lots of sizes $Q_1$ and $Q_2$ where $Q_1 \neq Q_2$. The holding cost in this policy is:

$$\tfrac{1}{2}Q_1(Q_1/D_1) + \tfrac{1}{2}Q_2(Q_2/D_1) = \tfrac{1}{2}(Q_1^2+Q_2^2)/D_1.$$

Look at the revised policy where we produce two lots of size $(Q_1+Q_2)/2$. This policy certainly is feasible if the original policy is feasible. The cost of this policy is:

$$2(\tfrac{1}{2}((Q_1+Q_2)/2)(((Q_1+Q_2)/2)/D_1)) = ((Q_1+Q_2)/2)^2/D_1.$$

So if $\left(\frac{Q_1+Q_2}{2}\right)^2 < \frac{Q_1^2+Q_2^2}{2}$ then the revised policy is cheaper than the original. This is clearly the case. □

In a renewal policy, the warehouse produces at every renewal point by definition. Furthermore, suppose that at every warehouse production point, we also have a renewal. This special type of renewal policy is referred to as a single cycle policy. The best single cycle policy is clearly stationary at the retailers and at the warehouse.

The optimal single cycle policy may be found by solving:

$$\text{Minimize } z = \sum_{j=0}^{N} \left( \frac{n_j K_j}{T} + \frac{h_j T D_j}{2n_j} \right), \tag{2.2}$$

$$\text{subject to } T \geq 0, \tag{2.3}$$

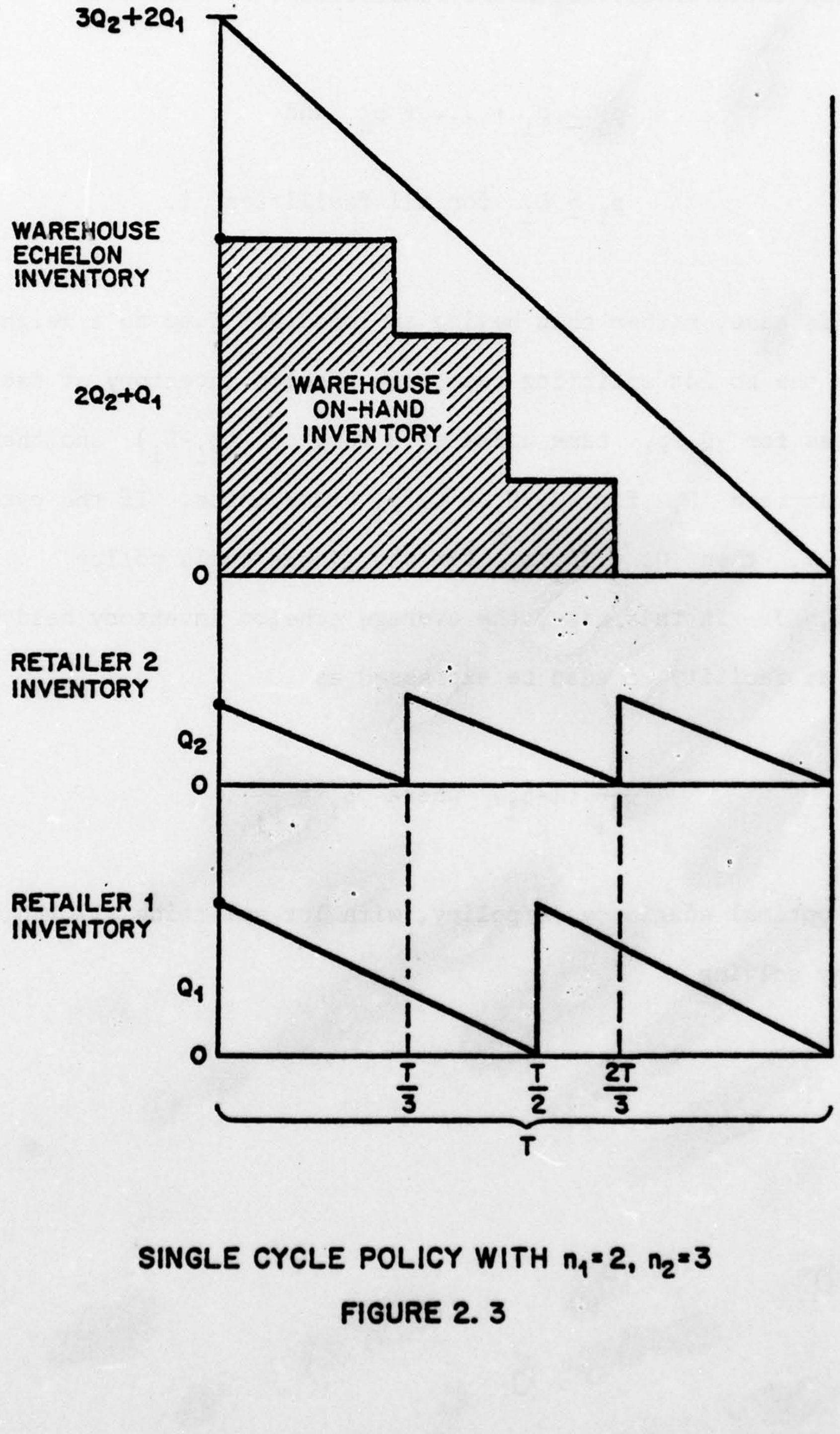
$$n_0 = 1, \tag{2.4}$$

$$n_j \geq 1, \text{ integer for } j = 1,\ldots,N, \tag{2.5}$$

where $n_j$, $j \geq 1$, is defined as the integer number of lots delivered to stage $j$ from the warehouse and $T$ is the length of time between renewals.

The equation (2.2) is the expression for the average cost per unit time for the single cycle policy $(n_1,n_2,\ldots,n_N)$. Clearly, if the warehouse sets up every $T$ time units, the average set up cost per unit time is $K_0/T$. Retailer $i$ sets-up $n_i$ times every $T$ time units so the average set-up cost per unit time for retailer $i$ is $n_i K_i/T$. If retailer $i$ produces a lot of size $Q_i$ at each production, then because of the constant demand rate $D_i$, the retailer holds $\frac{Q_i}{2}$ units on the average. The total demand at retailer $i$ during the cycle length $T$ is $T \cdot D_i$. Since $n_i$ equal lots are produced by Lemma 2.1, $Q_i = \frac{T \cdot D_i}{n_i}$. The average holding cost per unit time at retailer $i$ can therefore be written as $h_i T \cdot D_i / 2n_i$. The warehouse holding cost can be expressed similarly because the echelon inventory is drawn down at the constant rate, $D_0$. The validity of using the average cost per unit time formulation has been verified by Schwarz (1973).

In Figure 2.3, a two retailer single cycle policy is illustrated where $n_1 = 2$ and $n_2 = 3$.




SINGLE CYCLE POLICY WITH $n_1 = 2$, $n_2 = 3$

FIGURE 2. 3

Let us consider the implications of relaxing our assumption that no lot splitting is permitted. In this case, we can also assume a finite production rate $p_i$ at each facility $i = 0,1,\ldots,N$. If we start with zero initial inventories, to insure feasibility, we must have

$$p_0 \geq p_1 + \cdots + p_N \text{ and}$$

$$p_i \geq D_i \text{ for all facilities } i.$$

In this case, rather than having an immediate jump to a height of $Q_i$, as in the no lot splitting case, the echelon inventory at facility $i$ increases for $Q_i/p_i$ time units at a rate of $(p_i-D_i)$ and then decreases at rate $D_i$ for $Q_i/D_i - Q_i/p_i$ time units. If the cycle length is $T$, then $Q_i = TD_i/n_i$ for the single cycle policy $(n_1,n_2,\ldots,n_N)$. In this case, the average echelon inventory held per unit time at facility $i$ can be expressed as

$$\frac{TD_i}{2n_i}(1-\rho_i) \text{ where } \rho_i = \frac{D_i}{p_i}.$$

Thus, the optimal single cycle policy, with lot splitting allowed, may be found by solving

$$\text{Minimize} \quad z = \sum_{j=0}^{N} \left( \frac{n_j K_j}{T} + \frac{h_j T D_j}{2n_j} (1-\rho_j) \right)$$

$$\text{subject to} \quad T \geq 0,$$

$$n_0 = 1,$$

$$n_j \geq 1, \quad \text{integer for } j = 1,\ldots,N,$$

$$\text{and} \quad \rho_i = \frac{D_i}{P_i}.$$

Since $\rho_i$ is a constant we can define

$$h_j' = h_j(1-\rho_j)$$

and the optimization problem above reduces to precisely the one described by (2.2)-(2.5), that is, finding the optimal single cycle policy with no lot splitting permitted.

The following results concerning single cycle policies have been established (Schwarz, 1973).

Theorem 2.1. For the one warehouse, one retailer system, the optimal policy is a single cycle policy. □

Theorem 2.2. For the one warehouse, N retailer system, where the N retailers have identical costs and demand rates, the optimal policy is a single cycle policy. □

We can derive more general relationships concerning the optimal single cycle policies and the retailer costs.

Theorem 2.3. If $h_iD_i/K_i < h_jD_j/K_j$, then $n_i^* \le n_j^*$ in the optimal single cycle policy.

Proof of Theorem 2.3.

Suppose we have the optimal single cycle solution $(n_1^*, n_2^*, \ldots, n_N^*)$ and an optimal cycle length $T^*$ such that $h_1D_1/K_1 < h_2D_2/K_2$ but $n_1^* > n_2^*$. The cost of this policy is less than or equal to that for all policies with the same cycle length but a different vector of $n_i$'s. In particular, the cost of the optimal policy is smaller than or equal to the cost of the solutions $(n_1^*, n_1^*, n_3^*, \ldots, n_N^*)$ and $(n_2^*, n_2^*, n_3^*, \ldots, n_N^*)$ both with cycle length $T^*$. If we were to write out the cost function (2.2) for these 3 plans with cycle length $T^*$, the only terms that differ involve retailers 1 and 2. Thus

$$\frac{n_1^*K_1}{T^*} + \frac{n_2^*K_2}{T^*} + \frac{T^*h_1D_1}{2n_1^*} + \frac{T^*h_2D_2}{2n_2^*} \le \frac{n_1^*K_1 + n_1^*K_2}{T^*} + \frac{T^*(h_1D_1 + h_2D_2)}{2n_1^*} \quad (2.6)$$

and

$$\frac{n_1^*K_1}{T^*} + \frac{n_2^*K_2}{T^*} + \frac{T^*h_1D_1}{2n_1^*} + \frac{T^*h_2D_2}{2n_2^*} \le \frac{n_2^*K_1 + n_2^*K_2}{T^*} + \frac{T^*(h_1D_1 + h_2D_2)}{2n_2^*} . \quad (2.7)$$

Thus

$$\frac{n_2^*K_2}{T^*} + \frac{T^*h_2D_2}{2n_2^*} \le \frac{n_1^*K_2}{T^*} + \frac{T^*h_2D_2}{2n_1^*}$$

and

$$\frac{n_1^*K_1}{T^*} + \frac{T^*h_1D_1}{2n_1^*} \le \frac{n_2^*K_1}{T^*} + \frac{T^*h_1D_1}{2n_2^*} .$$

Thus

$$\frac{T^*h_2D_2}{2}\left(\frac{n_1^*-n_2^*}{n_2^*n_1^*}\right) \le \frac{K_2}{T^*}\,(n_1^*-n_2^*)$$

and

$$(n_1^*-n_2^*)\frac{K_1}{T^*} \le \frac{T^*h_1D_1}{2}\left(\frac{n_1^*-n_2^*}{n_2^*n_1^*}\right).$$

Dividing both sides of both inequalities by $n_1^* - n_2^* > 0$ we get:

$$\frac{h_2D_2}{K_2} \le \frac{2n_2^*n_1^*}{(T^*)^2} \le \frac{h_1D_1}{K_1}.$$

This contradicts our original assumption that $\frac{h_2D_2}{K_2} > \frac{h_1D_1}{K_1}$. □

<u>Theorem 2.4</u>. If $h_iD_i/K_i = h_jD_j/K_j$, then $n_i^* = n_j^*$ in an optimal single cycle policy.

<u>Proof of Theorem 2.4</u>.

Suppose we have the optimal single cycle solution $(n_1^*,n_2^*,\ldots,n_N^*)$ and an optimal cycle length $T^*$ such that $h_1D_1/K_1 = h_2D_2/K_2$ but $n_1^* \neq n_2^*$. The cost of this policy is less than or equal to that for all policies with the same cycle length but a different vector of $n_i$'s and as in the proof of Theorem 2.3 we again arrive at equations (2.6) and (2.7). Thus

$$\frac{n_2^*K_2}{T^*} + \frac{T^*h_2D_2}{2n_2^*} \le \frac{n_1^*K_2}{T^*} + \frac{T^*h_2D_2}{2n_1^*} \tag{2.8}$$

and

$$\frac{n_1^*K_1}{T^*} + \frac{T^*h_1D_1}{2n_1^*} \le \frac{n_2^*K_1}{T^*} + \frac{T^*h_1D_1}{2n_2^*}. \tag{2.9}$$

Rewriting (2.8) we have

$$(n_2^*-n_1^*) \le (T^*)^2\left(\frac{h_2D_2}{2K_2}\right)\left(\frac{1}{n_1^*}-\frac{1}{n_2^*}\right). \qquad (2.10)$$

Since by assumption $h_2D_2/K_2 = h_1D_1/K_1$, (2.10) becomes

$$(n_2^*-n_1^*) \le (T^*)^2\left(\frac{h_1D_1}{2K_1}\right)\left(\frac{1}{n_1^*}-\frac{1}{n_2^*}\right). \qquad (2.11)$$

By (2.9)

$$(T^*)^2\left(\frac{h_1D_1}{2K_1}\right)\left(\frac{1}{n_1^*}-\frac{1}{n_2^*}\right) \le (n_2^*-n_1^*). \qquad (2.12)$$

So clearly for (2.11) and (2.12) to hold simultaneously, both relations must be equalities which implies that (2.8) and (2.9) are also equalities. Then clearly we can change the optimal policy from $(n_1^*,n_2^*,n_3^*,\ldots,n_N^*)$ to either $(n_1^*,n_1^*,n_3^*,\ldots,n_N^*)$ or $(n_2^*,n_2^*,n_3^*,\ldots,n_N^*)$ without any increase in cost. □

## 2.5 Optimal Cycle Length and Separate Retailing

We will begin by examining the cost for a single cycle policy in greater detail. Recall (2.2)-(2.5)

$$\text{Min } z = \sum_{j=0}^{N}\left(\frac{n_jK_j}{T} + \frac{h_jTD_j}{2n_j}\right),$$

$$T \ge 0,$$

$$n_0 = 1,$$

$$n_j \ge 1, \quad j = 1,\ldots,N.$$

For fixed values of $n_j$, $j = 1,\dots,N$, the optimal $T$, denoted $T^*(n_j)$, satisfies:

$$T^*(n_j) = \left[ \frac{2 \sum_{j=0}^{N} n_j K_j}{\sum_{j=0}^{N} h_j D_j / n_j} \right]^{1/2} \tag{2.13}$$

and the optimal average cost per unit time denoted $C^*(n_j)$, satisfies:

$$C^*(n_j) = [2(\sum_{j=0}^{N} n_j K_j)(\sum_{j=0}^{N} h_j D_j/n_j)]^{1/2} . \tag{2.14}$$

Since (2.2) is separable and convex in the $n_j$, we may determine the optimal $n_j$, $j = 1,\dots,N$, for fixed $T$, denoted $n_j^*(T)$, as the smallest $n$ satisfying

$$n_j(n_j+1) \geq T^2(h_j D_j/2K_j). \tag{2.15}$$

Given the value $C^*(n_j)$ as in (2.14), it is possible to show that if two problems have the same basic cost structure, then the optimal solution vector of $n_i$'s will be the same. More specifically, we can state

<u>Theorem 2.5</u>. If we have 2 different one warehouse, $N$ retailer problems with costs $K_i, h_i, D_i$ and $K_i', h_i', D_i'$, $i = 0,\dots,N$ where

$$K_i' = \alpha K_i \qquad i = 0,\dots,N$$

and

$$h_i' \cdot D_i' = \beta \cdot h_i \cdot D_i \qquad i = 0,\dots,N$$

then the optimal single cycle solution vectors $(n_1^*,\ldots,n_N^*)$ and $(n_1^{*\prime},\ldots,n_N^{*\prime})$ are identical.

Proof of Theorem 2.5.

Suppose $(n_1^*,\ldots,n_N^*)$ is the optimal solution for the first problem. Then

$$C(n_1^*,\ldots,n_N^*) \leq C(n_1,\ldots,n_N) \quad \text{for } n_0 = 1 \text{ and for all}$$

$$n_1,\ldots,n_N, \text{ integer}, \geq 1 \qquad (2.16)$$

and by (2.14):

$$[2(\sum_{j=0}^{N} n_j^* K_j)(\sum_{j=0}^{N} h_j D_j/n_j^*)]^{1/2} \leq [2(\sum_{j=0}^{N} n_j K_j)(\sum_{j=0}^{N} h_j D_j/n_j)]^{1/2}$$

$$\text{for } n_0 = 1 \text{ and for all}$$

$$n_1,\ldots,n_N, \text{ integer}, \geq 1. \qquad (2.17)$$

Multiply both sides of (2.17) by $(\alpha\beta)^{1/2}$. Then regrouping terms yields

$$[2(\sum_{j=0}^{N} n_j^* \alpha K_j)(\sum_{j=0}^{N} \beta h_j D_j/n_j^*)]^{1/2} \leq [2(\sum_{j=0}^{N} n_j \alpha K_j)(\sum_{j=0}^{N} \beta h_j D_j/n_j)]^{1/2}$$

$$\text{for } n_0 = 1 \text{ and for all}$$

$$n_1,\ldots,n_N, \text{ integer}, \geq 1.$$

This clearly demonstrates that $(n_1^*,\ldots,n_N^*)$ is also the optimal solution for the problem with costs $K_i', h_i', D_i'$.

These two problems do not have identical ratios but the relationships between these ratios within each problem are the same. □

We will now examine the one warehouse, one retailer problem more completely. For one retailer, i.e. $N = 1$, $D_0 = D_1$ and (2.14) becomes

$$C^*(n_1) = [2(K_0+n_1K_1)(h_0D_1+h_1D_1/n_1)]^{1/2}. \tag{2.18}$$

Given the convexity of the expression within the brackets of (2.11), the optimal $n_1$, denoted $n_1^*$, is chosen as the smallest integer satisfying:

$$n(n+1) \geq K_0h_1/K_1h_0. \tag{2.19}$$

Thus we can explicitly solve the one warehouse, one retailer case to find the best single cycle policy, which by Theorem 2.1 is the overall optimal policy.

This suggests an alternative policy for the one warehouse, $N$ retailer system. If we split the one warehouse, $N$ retailer system into $N$ one warehouse, one retailer systems and solve each of these one retailer systems separately, we will arrive at the policy called <u>separate retailing</u>.

It is clear that the separate retailing policy is the overall optimal solution for the special case where $K_0 = 0$. If $K_0 = 0$, then the on-hand inventory at the warehouse is always zero in the overall optimal solution. Therefore, the warehouse echelon holding cost, $h_0$, is only charged against the inventory held at the retailers. In this case, we can solve $N$ independent single facility problems with the $i^{th}$ such problem having a set-up cost of $K_i$ and a holding cost of $(h_i+h_0)$, and then schedule set-ups at the warehouse whenever they are necessary. By setting $K_0$ to zero in (2.18) and (2.19) it is clear that the procedure outlined above is simply separate retailing.

## 2.6 An Algorithm to Find the Optimal Single Cycle Policy

In this section, we will present a method that is guaranteed to find the optimal single cycle policy. Goyal (1974) has presented an algorithm that determines the optimum packaging frequency of items jointly replenished. A number of items are manufactured jointly and then packaged with different frequencies. Goyal's method determines an $\overline{N}$, the frequency of manufacturing set-ups for the product, and a $\overline{K}_j$, the frequency of manufacturing set-ups for the product/frequency of packaging set-ups, to minimize

$$C(\overline{N},\overline{K}_j) = \overline{N}(\overline{S} + \sum_{j=0}^{\overline{n}} \overline{S}_j/\overline{K}_j) + \frac{1}{2\overline{N}} \sum_{j=0}^{\overline{n}} \overline{h}_j \cdot \overline{Q}_j \overline{K}_j \qquad (2.20)$$

where $\overline{n}+1$ = number of items,

$\overline{S}$ = manufacturing set-up cost,

$\overline{Q}_j$ = demand per year,

$\overline{h}_j$ = stock holding cost per unit per year,

and $\overline{S}_j$ = cost of a packaging set-up.

It should be noted that $\overline{N}$ is continuous and $\overline{K}_j$ is discrete. If we redefine,

$$\overline{N} = T,\ \overline{S} = 0,\ \overline{S}_j = h_j D_j/2,\ \overline{K}_j = n_j,\ \overline{h}_j \overline{Q}_j/2 = K_j \text{ and } \overline{n} = N$$

then (2.20) becomes

$$C(T,n_j) = T(\sum_{j=0}^{N} h_jD_j/2n_j) + \frac{1}{T}(\sum_{j=0}^{N} K_j\cdot n_j)$$

$$= \sum_{j=0}^{N} (n_jK_j/T + h_jD_jT/2n_j). \tag{2.21}$$

This is simply the cost of the single cycle policy given in (2.2) provided $n_0 = 1$. Thus, the algorithm presented by Goyal, with suitable redefinition of terms, will find the optimal single cycle policy.

In order to understand Goyal's method, we will recall from (2.15) that given $T$, the best $n_j$, denoted $n_j^*(T)$, has the property that

$$n_i^*(T)(n_i^*(T) - 1) < \frac{T^2h_iD_i}{2K_i} \leq n_i^*(T)(n_i^*(T) + 1) \tag{2.22}$$

Thus,

$$\eta_i(n_i^*(T)(n_i^*(T) - 1))^{1/2} < T \leq \eta_i(n_i^*(T)(n_i^*(T) + 1))^{1/2} \tag{2.23}$$

where

$$\eta_i = (2K_i/h_iD_i)^{1/2}. \tag{2.24}$$

For each retailer $i$, we can summarize this information as in Table 2.1 which gives the range of the cycle length $T$ for which $n_i(T)$ remains constant. We can put all these tables together to form one master list of intervals, $I_1, I_2, I_3, \ldots$ where the values of $n_j$ remain unchanged in each interval. We can then compute the cost of each permissible solution by going through this master list interval by interval each time comparing the cost to the best value found so far. Since at each step we only change a single $n_j$ by one, we will eventually hit upon the optimal solution.

| T | $n_i(T) = 1$ | $n_i(T) = 2$ | $n_i(T) = 3$ | $n_i(T) = 4$ |
|---|---|---|---|---|
| Lower bound | 0 | $\eta_i(2)^{1/2}$ | $\eta_i(6)^{1/2}$ | $\eta_i(20)^{1/2}$ |
| Upper bound | $\eta_i(2)^{1/2}$ | $\eta_i(6)^{1/2}$ | $\eta_i(12)^{1/2}$ | $\eta_i(30)^{1/2}$ |

Lower and Upper Bounds for $T$ for Different Values of $n_i(T)$

Table 2.1

It is clear from (2.13) that $T^*(n_j)$ increases as the $n_j$ increase. In particular,

$$T(1,1,\ldots,1) < T(n_1,n_2,\ldots,n_N)$$

if at least one $n_i > 1$ and all the other $n_j \geq 1$. Thus we can begin the Goyal procedure with cycle length $T(1,1,\ldots,1)$. To halt the procedure we will use an upper bound on the value of $T^*$. Clearly,

$$C(n_1^*,\ldots,n_N^*) = \frac{K_0}{T^*} + \frac{T^*h_0D_0}{2} + \sum_{i=1}^{N} \frac{n_i^*K_i}{T^*} + \frac{T^*h_iD_i}{2n_i^*}$$

$$> \frac{K_0}{T^*} + \frac{T^*h_0D_0}{2} + \sum_{i=1}^{N} (2K_ih_iD_i)^{1/2}. \qquad (2.25)$$

This is the case because if we relax the integer restriction on the $n_i$, then

$$n_i(T) = T(h_iD_i/2K_i)^{1/2} = T/\eta_i \qquad (2.26)$$

where $\eta_i$ is defined as in (2.24), and we can substitute this value for $n_i(T)$ into the cost expression and get $(2K_ih_iD_i)^{1/2}$. In addition,

$$C(1,1,\ldots,1) \geq C(n_1^*,\ldots,n_N^*)$$

so

$$C(1,1,\ldots,1) > K_0/T^* + T^*h_0D_0/2 + \sum_{i=1}^{N} (2K_ih_iD_i)^{1/2}. \qquad (2.27)$$

Define

$$\Delta(1,1,1,\ldots,1) = C(1,1,\ldots,1) - \sum_{i=1}^{N} (2K_ih_iD_i)^{1/2}. \qquad (2.28)$$

Thus

$$K_0/T^* + T^*h_0D_0/2 < \Delta(1,1,\ldots,1). \qquad (2.29)$$

The left hand side of (2.28) is convex in $T^*$ and therefore

$$T^* < \overline{T}(1,1,\ldots,1)$$

where

$$\overline{T}(1,1,\ldots,1) = \frac{\Delta(1,1,\ldots,1) + [(\Delta(1,1,\ldots,1))^2 - 2K_0h_0D_0]^{1/2}}{h_0D_0}. \qquad (2.30)$$

This bound on $T^*$ can be used to halt the search procedure on the $n_i$'s.

It should be pointed out that Silver (1975) has presented a simple graphical procedure to solve the problem presented by Goyal. This same procedure is not applicable to the single cycle problem because of the restriction that the warehouse only produces once in each cycle

## 2.7 A Heuristic Algorithm to Find Good Single Cycle Policies

The Goyal algorithm, presented in the preceding section, increases only one $n_j$ by 1 at each iteration and at each iteration a comparison is made to the incumbent solution. In the heuristic method, we will reduce the number of comparisons to the incumbent by allowing the $n_i$ to increase more quickly if possible. Given a set of $n_i$'s we can compute the best cycle length for those $n_i$'s, denoted $T^*(n_i)$, using the formula in (2.13) and given the cycle length $T$, we can compute the best unrestricted $n_i(T)$ using (2.26). The heuristic is essentially an iterative procedure that alternates between computing the $T^*(n_i)$ and the $n_i(T)$. Since the minimum cost given a vector of $n_i$'s, denoted $C^*(n_i)$ and computed as in (2.14), is not a convex function, we could converge upon a local minimum by simply iterating back and forth between $T^*(n_i)$ and $n_i(T)$. We will avoid this problem by insuring that the cycle length is increased at each iteration which may force us to move in a direction that does not decrease the cost, $C^*(n_i)$.

As in the Goyal method, we will begin by evaluating $T(1,1,\ldots,1)$ and $C(1,1,\ldots,1)$. We can compute the upper bound, $\overline{T}(1,1,\ldots,1)$, on $T^*$ using (2.30) as in the Goyal algorithm. We will use $T(1,1,\ldots,1)$ to compute the best unrestricted $n_i(T(1,1,\ldots,1)) = T(1,1,\ldots,1)\left[\frac{h_i D_i}{2K_i}\right]^{1/2}$ for all $i$. We will examine all the lattice points surrounding this possibly non-integral vector of $n_i$'s. By examining all lattice points surrounding the unrestricted $n_i$'s, the $n_i^*(T)$ as defined by (2.15) are included. We will, however, only compute the cost for a vector of integer $n_i$'s provided

$$n_i \geq 1 \qquad i = 1,\ldots,N,$$

$$n_j > 1 \qquad \text{for some } 1 \leq n_j \leq N,$$

$$n_i \leq n_j \qquad \text{if} \quad h_iD_i/K_i < h_jD_j/K_j,$$

and
$$n_i = n_j \qquad \text{if} \quad h_iD_i/K_i = h_jD_j/K_j.$$

Clearly, we need only check solutions that satisfy Theorems 2.3 and 2.4. All the $n_i \geq 1$ from the original statement of the problem and there is no point in computing the cost of $(1,1,\ldots,1)$ again.

Let

$$\overline{n}_i = \text{smallest integer greater than or equal to } n_i(T(1,1,\ldots,1)).$$

We can compute $T(\overline{n}_1,\overline{n}_2,\ldots,\overline{n}_N)$ and check if this cycle length exceeds the upper bound on $T^*$ determined using (2.30). If it does not exceed the upper bound, we will compute the best unrestricted $n_i$'s based on this larger cycle length and repeat the procedure outlined above. If it does exceed the upper bound, we will halt the procedure. It is also possible that when we generate the new unrestricted $n_i$'s that all the surrounding lattice points have already been examined. In this case, we will halt the procedure as well. To improve the bounding procedure, we can recompute the upper bound on $T^*$ whenever we find a better incumbent solution by computing $\overline{T}^*$(Incumbent) using (2.30).

To demonstrate the heuristic procedure, let us examine the following example:

$$K_0 = .1, \quad h_0 = 1, \quad D_0 = 2,$$

$$K_1 = 99.9, \quad h_1 = 99, \quad D_1 = 1, \tag{2.31}$$

$$K_2 = 99.9, \quad h_2 = 199, \quad \text{and} \quad D_2 = 1.$$

To begin, we will compute $1/\eta_i$ for all $i$, where $\eta_i$ defined as in (2.24). That is, $1/\eta_i = [h_i D_i/2K_i]^{1/2}$. From Theorem 2.3, we know if $1/\eta_1 < 1/\eta_2$, then $n_1^* \leq n_2^*$. For this example $1/\eta_1 = .7039$ and $1/\eta_2 = .9980$ so we need only examine those lattice points for which $n_1 \leq n_2$. Next we compute

$$C(1,1) = 346.323, \quad T(1,1) = 1.15 \quad \text{and} \quad \overline{T}(1,1) = 6.266.$$

Next we compute the best unrestricted $(n_1,n_2)$ for $T(1,1)$:

$$n_1(T(1,1)) = 1/\eta_1 \cdot T(1,1) = .8126 \quad \text{and} \quad n_2(T(1,1)) = 1/\eta_2 \cdot T(1,1) = 1.15.$$

The only lattice point to be examined is (1,2) since (1,1) has already been checked and

$$C(1,2) = 346.727, \quad T(1,2) = 1.729.$$

Since $T(1,2) < \overline{T}(1,1)$, we continue to the next iteration. We compute the best unrestricted $(n_1,n_2)$ for $T(1,2)$:

$$n_1(T(1,2)) = 1.21 \quad \text{and} \quad n_2(T(1,2)) = 1.72.$$

The lattice points surrounding (1.21, 1.72) are (1,1), (1,2), (2,1) and (2,2). The first two of these four lattice points have already been examined and the third can be eliminated since $n_1 > n_2$ and by Theorem 2.3 this is impossible.

The only lattice point to be examined is (2,2) and

$$C(2,2) = 347.432, \quad T(2,2) = 2.30 < \overline{T}(1,1).$$

We compute the best unrestricted $(n_1,n_2)$ for $T(2,2)$:

$$n_1(T(2,2)) = 1.61 \quad \text{and} \quad n_2(T(2,2)) = 2.29.$$

The lattice points surrounding (1.61, 2.29) are (1,2), (2,2), (1,3) and (2,3). Only the last two lattice points have not yet been examined. Therefore we compute

$$C(1,3) = 365 \quad \text{and} \quad C(2,3) = 343.13125.$$

Clearly $C(2,3) < C(1,1)$ and (2,3) becomes the new incumbent solution. We can compute $\overline{T}(2,3) < \overline{T}(1,1)$ using (2.30) and

$$\overline{T}(2,3) = 3.056.$$

To insure that $(n_1,n_2)$ increase the most at the next iteration we next compute $T(2,3)$ since (2,3) is the largest lattice point neighboring the non-integer $n_i$'s from the previous iteration. We therefore compute

$$T(2,3) = 2.912 < \overline{T}(2,3)$$

and we continue the algorithm by computing

$$n_1(T(2,3)) = 2.05 \quad \text{and} \quad n_2(T(2,3)) = 2.91.$$

The only previously unexamined lattice points are (3,2) and (3,3). (3,2) can be eliminated by Theorem 2.3 and so we need only compute

$$C(3,3) = 346.84 \quad \text{and} \quad T(3,3) = 3.457.$$

$T(3,3) > \overline{T}(2,3)$ and so we can halt the procedure. The best solution found by the heuristic is (2,3) and, in fact, this is the optimal single cycle policy for this problem.

It should be pointed out that although the example presented above has a local minimum at (1,1), the heuristic procedure does not get "caught" by it. This is due to the fact that the heuristic does not have to move in a direction that decreases cost. Regardless of the costs, the heuristic increases the cycle length at each iteration.

## 2.8 Computational Comparisons

The Goyal and heuristic algorithms have both been programmed in order to assess the quality of the heuristic and possible differences in computational effort. The problems ranged in size from 2 to 7 retailers, the holding and set-up costs were uniformly distributed on the integers from 1 to 100 and the demand rates were uniformly distributed on the integers from 1 to 10. The measure of effectiveness is the number of times the algorithm computes the cost of a complete plan and compares that cost to the value of the incumbent solution. To keep the two

procedures on an equal footing, the updating of the upper bound on $T^*$ discussed for the heuristic is not used.

On the average, the heuristic procedure performed better than the Goyal algorithm. For the 140 test problems, the Goyal algorithm averaged 8.32 comparisons to the incumbent the heuristic averaged 6.61 comparisons per problem. More importantly, the heuristic found the optimal solution in all 140 problems. There is, however, one disturbing factor. In 22% of the problems, the heuristic procedure required more comparisons than the Goyal algorithm. In the worst case, the Goyal algorithm required 119 comparisons and the heuristic 150. This might seem contradictory, since the heuristic was constructed in such a way as to allow the vector of $n_i$'s to increase more quickly at each iteration than in the Goyal algorithm. The problem with the heuristic lies in the examination of all the lattice points surrounding the vector of non-integer $n_i$'s. For certain problems, this can lead to a large number of unnecessary comparisons when using the heuristic procedure. As the number of retailers increases, the number of lattice points to be examined at each iteration in the heuristic increases exponentially. The performance of the heuristic therefore declines in relation to the Goyal algorithm for larger problems. Fortunately, a simple selection rule can improve the computational efficiency of the heuristic significantly without greatly hurting its performance in terms of finding the optimal solution. This improved heuristic will be discussed in the next section.

## 2.9 An Improved Heuristic Algorithm to Find Good Single Cycle Policies

In this section, we will discuss a method to revise the heuristic algorithm presented in Section 2.7. Essentially, instead of examining all the lattice points surrounding the non-integer $n_i$ vector, we will examine only two particular lattice points. Thus the maximum number of solutions to be evaluated and compared to the incumbent at each iteration is two, regardless of the number of retailers in the problem.

We would like to ensure that the revised version of the heuristic examines fewer solutions than the original heuristic. Since the maximum number of solutions examined at each iteration in the revised heuristic is fewer than in the original approach, we need only guarantee that the number of iterations in the revised heuristic does not increase from the original. Recall that in the original heuristic, the largest lattice point neighboring the non-integer vector of $n_i$'s is the one which is used to get the next value of $T$, the cycle length. This same rule will be used in the revised heuristic and the increases in the cycle length at each step, and therefore the total number of iterations, will be the same in both heuristics.

It should also be noted that if the two algorithms have the same values for the cycle length, then the values for the best unrestricted $n_i$'s found using (2.26) at each iteration will also be identical. By examining a selected group of problems for which the original heuristic performed worse than the Goyal procedure, (Table 2.2) it was found that if in addition to examining the largest lattice point neighboring the non-integer vector of $n_i$'s we also examined the closest lattice point, then the optimal solution remained among those solutions examined. This rule

| | Set-Up Costs | Holding Costs | Demand Rate | Optimal Solution | Cost | # Comparisons Goyal | Heuristic |
|---|---|---|---|---|---|---|---|
| Warehouse | 24 | 27 | 18 | (1,1,1) | 816.9 | 1 | 2 |
| Retailer 1 | 98 | 36 | 9 | | | | |
| 2 | 71 | 81 | 6 | | | | |
| 3 | 29 | 69 | 3 | | | | |
| Warehouse | 72 | 35 | 23 | (1,1,2) | 838.4 | 3 | 4 |
| Retailer 1 | 92 | 4 | 10 | | | | |
| 2 | 25 | 28 | 5 | | | | |
| 3 | 35 | 93 | 8 | | | | |
| Warehouse | 65 | 95 | 26 | (1,1,2,3) | 1356.0 | 4 | 7 |
| Retailer 1 | 78 | 11 | 6 | | | | |
| 2 | 99 | 66 | 7 | | | | |
| 3 | 7 | 59 | 7 | | | | |
| 4 | 5 | 94 | 6 | | | | |
| Warehouse | 70 | 65 | 16 | (1,1,2,3) | 778.7 | 5 | 8 |
| Retailer 1 | 51 | 51 | 4 | | | | |
| 2 | 15 | 51 | 3 | | | | |
| 3 | 8 | 93 | 3 | | | | |
| 4 | 8 | 93 | 6 | | | | |
| Warehouse | 99 | 76 | 18 | (1,1,1,2) | 1184.9 | 2 | 3 |
| Retailer 1 | 63 | 43 | 4 | | | | |
| 2 | 57 | 93 | 6 | | | | |
| 3 | 31 | 95 | 6 | | | | |
| 4 | 10 | 61 | 4 | | | | |
| Warehouse | 76 | 69 | 16 | (1,1,1,2,2) | 924.2 | 3 | 6 |
| Retailer 1 | 62 | 44 | 1 | | | | |
| 2 | 50 | 98 | 1 | | | | |
| 3 | 19 | 25 | 8 | | | | |
| 4 | 10 | 88 | 2 | | | | |
| 5 | 13 | 77 | 4 | | | | |

Selected Examples: Heuristic Performs Worse than Goyal

Table 2.2

performed better than picking $n_i^*(T)$ using (2.15).

One can also mathematically justify the use of the closest lattice point. Let us consider a two retailer problem for the moment. Given a particular value for the cycle length $T$, the cost of the single cycle policy where retailer 1 produces $n_1$ times between each warehouse production and retailer 2 $n_2$ times can be written:

$$f(n_1,n_2|T) = \frac{K_0}{T} + \frac{Th_0D_0}{2} + \frac{n_1K_1}{T} + \frac{Th_1D_1}{2n_1} + \frac{n_2K_2}{T} + \frac{Th_2D_2}{2n_2} \tag{2.32}$$

and the gradient of $f(n_1,n_2|T)$ can be written

$$\nabla f(n_1,n_2|T) = \left(\frac{K_1}{T} - \frac{Th_1D_1}{2n_1^2}\ ;\ \frac{K_2}{T} - \frac{Th_2D_2}{2n_2^2}\right). \tag{2.33}$$

The unrestricted $(n_1^u,n_2^u)$ pair is chosen by (2.26) which is equivalent to setting $\nabla f(n_1^u,n_2^u|T) = 0$. Let us approximate the cost, given $T$, at the lattice point $(n_1^\ell,n_2^\ell)$ by expanding the cost function about $(n_1^u,n_2^u)$. That is:

$$\begin{aligned} f(n_1^\ell,n_2^\ell|T) &= f(n_1^u,n_2^u|T) + (n_1^\ell-n_1^u,\ n_2^\ell-n_2^u)^t\,\nabla f(n_1^u,n_2^u|T) \\ &\quad + \frac{1}{2}(n_1^\ell-n_1^u,\ n_2^\ell-n_2^u)\nabla^2 f(n_1^u,n_2^u|T)(n_1^\ell-n_1^u,\ n_2^\ell-n_2^u)^t. \end{aligned} \tag{2.34}$$

By definition, $\nabla f(n_1^u,n_2^u|T) = 0$, and from (2.33)

$$\nabla^2 f(n_1^u,n_2^u|T) = \begin{pmatrix} \dfrac{Th_1D_1}{(n_1^u)^3} & 0 \\ & \dfrac{Th_2D_2}{(n_2^u)^3} \end{pmatrix}. \tag{2.35}$$

Therefore, we can rewrite (2.34) as

$$f(n_1^\ell,n_2^\ell|T) = f(n_1^u,n_2^u|T) + \frac{1}{2}(n_1^\ell-n_1^u)^2\left(\frac{h_1D_1}{(n_1^u)^3}\right) + \frac{1}{2}(n_2^\ell-n_2^u)^2\left(\frac{h_2D_2}{(n_2^u)^3}\right) \quad (2.36)$$

and clearly the minimum functional increase in (2.35) over the value of $f(n_1^u,n_2^u|T)$ will take place at the closest lattice point to $(n_1^u,n_2^u)$. Therefore, we have demonstrated:

Lemma 2.2. If we have computed the best unrestricted $n_i(T)$, $i = 1,\ldots,N$, for a given value of $T$, then the closest lattice point to $(n_1(T),\ldots,n_N(T))$ will have the smallest increase in cost compared to the cost at $(n_1(T),\ldots,n_N(T))$. □

To summarize, the revised heuristic will examine only 2 lattice points neighboring the unrestricted $n_i(T)$'s at each iteration: the closest lattice point, denoted $(n_1^c(T),\ldots,n_N^c(T))$ and the largest lattice point, denoted $(\overline{n}_1(T),\ldots,\overline{n}_N(T))$. To review, let us rewrite the revised heuristic as follows:

Step 0: Set the active node equal to $(1,1,\ldots,1)$. Set the incumbent solution equal to $(1,1,\ldots,1)$ with cost $C^*(1,1,\ldots,1)$ using 2.9. Compute $\overline{T}$, the upper bound on $T^*$, using (2.28). Go to step 1.

Step 1: Compute $T = T^*$ (active node). If $T > \overline{T}$, halt. Otherwise go to step 2.

Step 2: Compute the best unrestricted $n_i(T)$ (insuring $n_i(T) \geq 1$) $i = 1,\ldots,N$ using (2.26). If $(n_i^c(T),\ldots,n_N^c(T))$, the closest lattice point, was not previously evaluated, then

compute the cost and compare it to the incumbent. Go to step 3.

Step 3: If $(\bar{n}_1(T),\ldots,\bar{n}_N(T))$, the largest lattice point, was not previously evaluated then compute its cost, compare this cost to the incumbent, set the new active node equal to $(\bar{n}_1(T),\ldots,\bar{n}_N(T))$, and go to step 1; otherwise, halt.

## 2.10 Computational Comparisons for Revised Heuristic

The decision rules in the revised heuristic were based in part upon the performance of the original heuristic on a particular series of test problems. To assess the performance of the revised heuristic adequately in comparison to the Goyal algorithm, it was necessary to generate a new series of 175 test problems. The problems ranged in size from 3 to 7 retailers, the set-up costs were uniformly distributed on the integers from 1 to 100, and the demand rates were uniformly distributed on the integers from 1 to 10. The holding costs were uniformly distributed on the integers from 1 to 10, from 1 to 100 and from 1 to 1000 for various groups of problems. The results for these 175 test problems are summarized in Table 2.3. In these 175 test problems, the Goyal algorithm averaged 8.43 comparisons to the incumbent per problem and the revised heuristic only 2.22 comparisons, a 73.6% improvement. In addition, even for these relatively small test problems, the revised heuristic solved the 175 problems in 3.16 seconds, the Goyal algorithm in 3.5 seconds. For these small problems, the average computation time is still very short for both procedures.

| No. of Retailers | No. of Problems | Holding Cost Range | # Comparisons Goyal | Heuristic | # Optimal |
|---|---|---|---|---|---|
| 3 | 25 | 1-10 | 109 | 31 | 25 |
| 3 | 25 | 1-100 | 121 | 31 | 25 |
| 3 | 25 | 1-1000 | 180 | 45 | 25 |
| 5 | 25 | 1-10 | 119 | 56 | 24 |
| 5 | 25 | 1-100 | 342 | 79 | 23 |
| 5 | 25 | 1-1000 | 300 | 63 | 24 |
| 7 | 25 | 1-1000 | 304 | 83 | 25 |
| Totals: | 175 | 1-1000 | 1475 | 388 | 171 |

Performance of Revised Heuristic vs. Goyal Algorithm

Table 2.3

The examination of only 2 lattice points has eliminated the problem of the original heuristic examining too many possible solutions in a significant proportion of the problems. In only 1 example out of 175 does the revised heuristic involve more comparisons than the Goyal procedure and the difference is only a single comparison. (For this one example the heuristic reevaluates a previously examined solution because it only checks this solution for repetition against those generated at the preceding iteration rather than 2 iterations back.)

Although the revised heuristic requires less computational effort than the Goyal approach, it no longer finds the optimal solution for every test problem. In four out of 175 cases (Table 2.4) or 2.3% the heuristic fails to find the optimal solution. Essentially, in these examples the heuristic is examining the wrong lattice point at a particular iteration. The average absolute error, for these four problems

| No. of Comparisons To Incumbent | | Cost of Final Incumbent | | Solution | | Absolute Error | % Error |
|---|---|---|---|---|---|---|---|
| Goyal | Revised Heur. | Goyal | Heur. | Goyal | Heur. | | |
| 20 | 6 | 771.9117 | 772.2543 | (1,1,1,4,6) | (1,1,1,3,6) | .3426 | .044% |
| 75 | 9 | 757.9408 | 758.6303 | (1,1,2,6,7) | (1,1,2,5,6) | .6895 | .091% |
| 11 | 6 | 3378.9765 | 3382.9260 | (1,1,1,1,4) | (1,1,1,1,3) | 3.9495 | .12% |
| 12 | 5 | 369.4374 | 379.0333 | (1,2,2,2,3) | (1,2,2,3,3) | .5959 | .16% |

Examples Where Revised Heuristic Fails to Find Optimum

Table 2.4

in error, is 1.39 and the average percentage error is only .1%, certainly small enough to be acceptable for all practical purposes.

The improved heuristic was also tested on a series of larger problems with real demand rates and holding costs for a large corporation. There were 20 retailers in these problems each with identical set-up and holding costs. The demand rates for retailers 2-20 were integer multiples of the demand rate at retailer 1 ranging from $2D_1$ to $100D_1$. The holding cost at the warehouse was 1.06 times the holding cost at the retailer and the set-up cost at the warehouse was varied to test the effectiveness of the two procedures. The results for the two algorithms are summarized in Table 2.5. Because of the similarity in cost structure, it is only necessary to examine selected examples (Theorem 2.5).

In all the examples tested, the heuristic found the optimal solution. More importantly, the heuristic seemed to remain as effective, both in terms of the number of comparisons and in time, as the set-up cost at

| $K_0$ | # Comparisons Goyal | # Comparisons Heuristic | Execution Time (seconds) Goyal | Execution Time (seconds) Heuristic |
|---|---|---|---|---|
| $2K_1$ | 7 | 5 | .07 | .07 |
| $10K_1$ | 7 | 4 | .06 | .07 |
| $100K_1$ | 40 | 8 | .13 | .08 |
| $1000K_1$ | 193 | 6 | >.41 | .08 |

Selected Computational Comparisons for Revised Heuristic vs. Goyal on 20 Retailer Problems

Table 2.5

the warehouse was increased relative to the set-up cost at the retailers. The Goyal procedure, on the other hand, became much worse than the revised heuristic, by either measure, as the set-up cost increased. As the set-up cost at the warehouse increases, the number of times the retailers produce within each cycle increases as well. The Goyal procedure only increases one $n_j$ by one at each iteration and the heuristic can make larger jumps and save a great deal of computational effort.

Graves and Schwarz (1977) have presented a branch and bound algorithm to find the optimal single cycle policy. Although no direct comparisons have been made, the improved heuristic seems superior for larger numbers of retailers. There have been no indications that the improved heuristic fails to find the optimal solution as frequently for problems with a larger number of retailers as was the case for the branch and bound algorithm of Graves and Schwarz. Although computation time seems comparable for the two approaches on small test problems, the enumeration tree in the branch and bound algorithm has a level for each retailer and

for extremely large problems, the storage requirements may be unwieldly. Also, at every level of the enumeration tree there is a branch for each integer value of $n_i$. Thus, for those problems where the $n_i$'s get large and the Goyal algorithm performs much worse than the revised heuristic, the enumeration tree in the branch and bound algorithm also will be wide at each level. Although no definitive statements can be made without further direct comparisons, the iterative heuristic seems at least as effective, if not more so for large problems, than the existing solution techniques.

## 2.11 Single Cycle Policies and Optimality

With the computational techniques discussed in sections (2.6)-(2.10), it is possible to determine the optimal single cycle policy. The major remaining question is whether a single cycle policy is the overall optimal policy. The answer to this question is yes, in the cases of one retailer or $N$ identical retailers from theorems 2.1 and 2.2, but we are interested in examining the correlation between single cycle policies and overall optimal policies in more general cases.

Let us begin by recalling the example used to demonstrate the heuristic algorithm:

$$K_0 = .1, \ h_0 = 1, \ D_0 = 2,$$
$$K_1 = 99.9, \ h_1 = 99, \ D_1 = 1, \qquad (2.31)$$
$$K_2 = 99.9, \ h_2 = 199, \text{ and } D_2 = 1.$$

For this example, the optimal single cycle policy has $(n_1,n_2) = (2,3)$

with an average cost per unit time of 343.13125. Let us split the problem into two independent one warehouse, one retailer problems, and then determine the best "separate retailing" policy as discussed in Section 2.5. First, we compute

$$K_0h_1/K_1h_0 = 9.9/99.9$$

and

$$K_0h_2/K_2h_0 = 19.9/99.9$$

and in both cases, the smallest integer $n$ such that $n(n+1)$ is greater than these quantities is one and thus best separate retailing policy has $(n_1^{SR}, n_2^{SR}) = (1,1)$ and the average cost per unit time of this policy from (2.18) is:

$$[2(K_0+1\cdot K_1)(h_0D_1+h_1D_1/1)]^{1/2} + [2(K_0+1\cdot K_2)(h_0D_2+h_2D_2/1)]^{1/2} = 341.42.$$

We have therefore demonstrated,

<u>Theorem 2.6</u>. The overall optimal policy may not be a single cycle policy and

<u>Corollary 2.7</u>. The best separate retailing policy may be less costly than the best single cycle policy.

In the best separate retailing policy described above, the retailers have optimal lot sizes of

$$Q_1^{SR} = T_1^{SR}\cdot D_1 = \left[\frac{2(K_0+K_1)D_1}{h_0+h_1}\right]^{1/2} = \sqrt{2} \quad \text{and}$$

$$Q_2^{SR} = T_2^{SR}\cdot D_2 = \left[\frac{2(K_0+K_2)D_2}{h_0+h_2}\right]^{1/2} = 1, \tag{2.37}$$

where $T_1^{SR}$ and $T_2^{SR}$ can be derived from (2.13) for one retailer. This separate retailing policy is stationary at the retailers but not at the warehouse. The warehouse produces lots of size $\sqrt{2}$ when it is producing for retailer 1 and lots of size 1 when it is producing for retailer 2.

Consider now the stationary policy based upon the separate retailing plan described above. The retailers again have lot sizes of $\sqrt{2}$ and 1; but, the warehouse produces a lot size of $1 + \sqrt{2}$ whenever its on-hand inventory is insufficient to meet a retailer's demand requirements. To evaluate the average cost per unit time for this stationary policy, it should be noted that, on the average the warehouse sets-up once every $(1 + \sqrt{2})/2$ time units and the average on-hand inventory per unit time at the warehouse is $(1 + \sqrt{2})/2$. Thus the average cost per unit time for this policy is:

$$C_s = \underbrace{.1\left(\frac{2}{1+\sqrt{2}}\right) + 1\left(\frac{1+\sqrt{2}}{2}\right)}_{\text{warehouse costs}} + \underbrace{\frac{99.9}{1} + \frac{99.9}{\sqrt{2}}}_{\substack{\text{retailer}\\ \text{set-up costs}}} + \underbrace{100\left(\frac{\sqrt{2}}{2}\right) + 200\left(\frac{1}{2}\right)}_{\substack{\text{retailer holding}\\ \text{costs}}}$$

$$= 342.54.$$

Thus this stationary policy, although more costly than the best separate retailing, is less costly than the best single cycle policy. □

The stationary policy described above cannot be a single cycle policy since there are no renewal points after time zero. (Due to the irrationality of $\sqrt{2}$, retailers 1 and 2 never produce simultaneously after time 0). This example therefore proves,

<u>Theorem 2.8</u>. Among the class of stationary policies, a single cycle policy need not be optimal. □

Theorem 2.8 contradicts Theorem 1 of Graves and Schwarz (1977) which states, that the optimal stationary policy for a one-warehouse, N retailer system is indeed a single cycle policy. Their proof of Theorem 1 is in error because it relies upon the application of properties P1 and P4 to the optimal stationary policy. These properties which hold for the overall optimal policies need not be valid for optimal stationary policies.

Even if we had not previously computed the cost of the best separate retailing policy, it is obvious that the stationary policy constructed above cannot be the overall optimal solution. The irrationality of $\sqrt{2}$, in addition to precluding the existence of renewal points as discussed above, also forces the warehouse to produce when its on-hand inventory is greater than zero. This violates property P1 which holds for all overall optimal policies. Graves and Schwarz (1978) have suggested that if we restrict our attention to the class of stationary policies which satisfy P1-P5, then among this class of policies, a single cycle policy is optimal. This, however, is also not the case and we can prove

<u>Theorem 2.9</u>. Among the class of stationary policies satisfying P1-P5, a single cycle policy need not be optimal.

<u>Proof of Theorem 2.9</u>.

The proof will be by construction of an example for which there

exists a stationary policy satisfying P1-P5 with lower cost than the best single cycle policy.

Consider the following one-warehouse, two retailer example:

$$K_0 = .1,\ \ h_0 = 1,\ \ D_0 = 1 + \sqrt{2},$$

$$K_1 = 34.9,\ \ h_1 = 70\sqrt{2} - 1,\ \ D_1 = \sqrt{2},$$

$$K_2 = 99.9,\ \ h_2 = 199,\ \ D_2 = 1. \tag{2.38}$$

For this example, one can determine that the best single cycle policy has $(n_1,n_2) = (3,2)$ with average cost per unit time equal to 300.38022. The best separate retailing policy for this problem has $(n_1^{SR},n_2^{SR}) = (1,1)$ and an average cost per unit time of 298.99494. The optimal lot sizes for this separate retailing policy are:

$$Q_1^{SR} = \left[\frac{2(K_0+K_1)D_1}{h_0+h_1}\right]^{1/2} = 1 \text{ and}$$

$$Q_2^{SR} = \left[\frac{2(K_0+K_2)D_2}{h_0+h_2}\right]^{1/2} = 1. \tag{2.39}$$

Therefore retailer 1 produces a lot of size 1 every $1/\sqrt{2}$ time units, retailer 2 produces a lot of size 1 every 1 time unit and the warehouse produces a lot of size 1 each time either of the retailers produce, except at time zero where the warehouse produces 2. Let us examine the stationary policy where the retailers follow the same policy as in separate retailing and the warehouse produces 2 each time it produces. The average time between productions at the

warehouse is $2/D_0 = 2/(1+\sqrt{2})$ and the average inventory held physically at the warehouse is $1/2$. The average cost per unit time for this stationary policy is:

$$C^s = \underbrace{.1\left(\frac{1+\sqrt{2}}{2}\right) + 1\left(\tfrac{1}{2}\right)}_{\text{warehouse costs}} + (34.9)(\sqrt{2}) + \underbrace{99.9 + 100}_{\text{retailer set-up costs}} + \underbrace{35\sqrt{2}}_{\text{retailer holding costs}}$$

$$= 299.37423.$$

Due to the irrationality of $\sqrt{2}$, there is no point at which retailer 1 and retailer 2 produce simultaneously after time 0. Thus there are no points after time zero where the system inventory is zero, hence no renewals. Due to the fact that both retailers have identical lot sizes, the warehouse will only produce when it has zero inventory on-hand. Thus, property P1 is satisfied for this stationary policy. Properties P2-P5 are obviously also satisfied and therefore this policy is stationary, satisfies P1-P5, and has lower cost than the best single cycle policy thereby establishing Theorem 2.9. □

As mentioned earlier, in the proof of their Theorem 1, Graves and Schwarz (1977) applied properties P1 and P4 to the optimal stationary policy which may not be valid. However, if we restrict our attention to the class of stationary policies satisfying properties P1 through P5, then the proof of Theorem 1 would seem to demonstrate that among this class of policies, a single cycle policy is optimal. Theorem 2.9 disproves this conjecture and therefore there must be an additional flaw in the arguments presented by Graves and Schwarz. The flaw lies in the

assertion that stationarity and property P4 together imply that there exists an integer $n_k$ such that every $n_k$th production run at the warehouse is a simultaneous production point for retailer $k$. Although property P4 implies that the time interval between simultaneous production points is finite, it is not necessarily the case that this time interval is also constant. In the example used to prove Theorem 2.9, the irrationality of $\sqrt{2}$ and the lack of renewal points causes the time interval between simultaneous production points to differ over time. Thus, there is no single integer $n_k$ such that every $n_k$th production point at the warehouse is also a production point for retailer $k$.

To summarize, we have demonstrated in Theorems 2.6, 2.8 and 2.9 that a single cycle policy need not be optimal for increasingly more restrictive classes of policies. A common property of all the examples presented in this section is the lack of renewal points in those policies that are less costly than the best single cycle policy. Further investigation of single cycle policies as a member of the class of renewal policies is warranted and this is the subject of the next section.

## 2.12 <u>Single Cycle Policies and Renewal Policies</u>

A single cycle policy is a special case of a renewal policy. In this section, we will derive results concerning the holding cost in multiple-cycling policies and then demonstrate that, in certain instances, single cycle policies are indeed optimal.

We begin by noting that the example used to prove Theorem 2.9 has a stationary, non-renewal policy that is less costly than the best single

cycle policy. The following theorem relates optimal, stationary and renewal policies.

Theorem 2.10. If the overall optimal policy is stationary, then it is a renewal policy.

Proof of Theorem 2.10.

We begin by assuming that the optimal policy is stationary and has no renewals. In this case, we will construct a non-stationary policy which has lower average cost per unit time than the hypothesized optimal policy.

By stationarity and property P1, retailer $i$ has a production point every $T_i = Q_i/D_i$ time units. By property P2, the warehouse produces only when at least one of the retailers produces and thus, without loss of generality, we may assume that the first warehouse production after time zero is a simultaneous production point for retailer 1; that is, the first warehouse production point after time zero is at time $k_1T_1$, $k_1$ integer. During $k_1T_1$, $\left\lceil \frac{k_1T_1}{T_j} \right\rceil$ lots of size $Q_j$, including the lot at time zero, are required for retailer $j$, where $\lceil x \rceil$ is the smallest integer greater than or equal to $x$. Let $k_j = \left\lceil \frac{k_1T_1}{T_j} \right\rceil$. $\frac{k_1T_1}{T_j}$ cannot be an integer for all $j \neq 1$, or $k_1T_1$ would be a renewal point, contradicting the hypothesis.

Since property P1 must be satisfied for the optimal solution, the warehouse must have zero inventory on-hand at $k_1T_1$. Therefore,

$$Q_0 = k_1Q_1 + \sum_{j=2}^{N} \left\lceil \frac{k_1T_1}{T_j} \right\rceil Q_j = k_1Q_1 + \sum_{j=2}^{N} k_jQ_j. \tag{2.41}$$

Since we have assumed that the first warehouse production occurs at $k_1T_1$, it is clear that

$$(k_j-1)T_j < k_1T_1 \leq k_jT_j \quad \text{for} \quad j \neq 1. \tag{2.42}$$

In fact, this inequality must be a strict inequality for at least one retailer or the system would have a renewal at time $k_1T_1$.

It is also clear from (2.41) and stationarity that the length of time between successive warehouse production points is greater than or equal to $\min\{k_1T_1, k_2T_2, \ldots, k_NT_N\}$, which is equal to $k_1T_1$ by (2.42). Thus, all warehouse production intervals have length at least $k_1T_1$.

Now consider those intervals where the on-hand inventory is zero at the warehouse. Due to the lack of renewals, intervals with lengths equal to infinitely many values between 0 and $\min(T_1, T_2, \ldots, T_N)$ may be found. Let

$$\delta = k_1T_1 \cdot \min_i(Q_1/(Q_0+Q_i)). \tag{2.43}$$

Look at an interval, $(p_1, p_2)$, with zero on-hand inventory at the warehouse beginning at time $p_1$ and lasting until time $p_2$ and length $\delta'$, $\delta' < \delta$. Let $p_0$ be the time of the warehouse production point immediately preceding $p_1$. It should be noted that $p_0$ and $p_2$ are warehouse production points, while $p_1$ is a production point for a retailer, or some group of retailers.

In the present policy, $Q_0$ units are produced at the warehouse at times $p_0$ and $p_2$, and zero units are produced at the warehouse at time $p_1$. At least one retailer produces at time $p_1$, and we will

denote the total produced, at this time, over all retailers by $\overline{Q}$. Clearly,

$$\overline{Q} \geq Q_j \quad \text{for some } j. \tag{2.44}$$

Consider a policy which is identical to the original optimal policy at all the retailers but is revised at the warehouse as follows:

(1) Produce $Q_0 - \overline{Q}$ at time $p_0$,

(2) Produce $Q_0 + \overline{Q}$ at time $p_1$, and

(3) Produce $0$ at time $p_2$.

The only cost differences from the original production plan are over the interval $(p_0, p_2)$. The set-up costs remain unchanged; but, in the revised policy, $\overline{Q}$ fewer units are held over the time interval $(p_0, p_1)$ and $Q_0$ additional units are held over the time interval $(p_1, p_2)$ as compared to the original plan. Therefore, the net change in inventory of this revision equals

$$Q_0(p_2 - p_1) - \overline{Q}(p_1 - p_0).$$

Let us assume that

$$\delta' = (p_2 - p_1) \geq (\overline{Q}/Q_0)(p_1 - p_0); \tag{2.45}$$

that is, the revised plan carries at least as much inventory at the warehouse as the original plan.

As discussed earlier, all warehouse production intervals have length at least $k_1 T_1$ and therefore

$$p_2 - p_0 \geq k_1 T_1. \tag{2.46}$$

But $p_2 - p_0 = (p_2 - p_1) + (p_1 - p_0) = \delta' + (p_1 - p_0)$. From (2.46),

$$p_1 - p_0 \geq k_1 T_1 - \delta'$$

and, therefore, (2.45) implies

$$\delta' \geq (\overline{Q}/Q_0)(k_1 T_1 - \delta'). \tag{2.47}$$

Since $\overline{Q} \geq Q_j$ for some $j$,

$$\delta' \geq (Q_j/Q_0)(k_1 T_1 - \delta') \quad \text{for some } j.$$

Rearranging terms, we see that

$$\delta' \left( \frac{Q_0 + Q_j}{Q_0} \right) \geq \frac{Q_j}{Q_0} (k_1 T_1) \quad \text{for some } j,$$

or $$\delta' \geq \left( \frac{Q_j}{Q_0 + Q_j} \right) k_1 T_1 \geq k_1 T_1 \cdot \min_i \left( \frac{Q_i}{Q_0 + Q_i} \right) = \delta \quad \text{by definition.}$$

This is clearly impossible by the choice of $\delta'$, which is strictly smaller than $\delta$. Therefore, this revised plan has fewer units of inventory carried, and hence lower cost, than the original over the time interval $(p_0, p_2)$.

By repeating this construction for all intervals with zero on-hand inventory at the warehouse and length less than $\delta$, we have demonstrated that there exists a policy with lower average cost per unit time than the hypothesized optimal policy. This establishes Theorem 2.10. □

Theorem 2.10 indicates that further examination of stationary renewal policies is in order. In stationary renewal policies, a renewal point will occur at every $n_0$th warehouse production point, where $n_0 \geq 1$ and integer. Clearly, if $n_0 = 1$, we have the usual single cycle policy. If $n_0 > 1$, we will call this policy a multiple cycle policy.

We will demonstrate that for a particular problem, the best single cycle policy may have higher average cost per unit time than a multiple cycle policy. This is not as obvious a fact as it may seem. Multiple cycle policies do constitute a wider range of policies than single cycle policies, since $n_0$ is not constrained to be 1. But the warehouse echelon inventory in a multiple cycle policy may not be easily computed.

To illustrate some of the properties of a multiple cycle policy, let us consider the multiple cycle policy with $(n_0,n_1,n_2)$ equal to $(2,2,3)$ and $T$ equal to the time between renewals. We begin by developing an expression to represent the average cost per unit time of this policy.

Retailer 1 produces lots of size $Q_1$ at times 0 and $T/2$. Retailer 2 produces lots of size $Q_2$ at times 0, $T/3$ and $2T/3$. In this multiple cycle policy, as in a single cycle policy, retailer 1 holds an average of $Q_1/2$ units per unit time, and retailer 2 holds an average of $Q_2/2$ units per unit time.

By stationarity, the warehouse produces two lots of size $Q_0$, where

$$Q_0 = \frac{TD_0}{n_0} = \frac{TD_1}{2} + \frac{TD_2}{2} = \frac{2Q_1}{2} + \frac{3Q_2}{2} . \tag{2.48}$$

If the warehouse was to produce at times 0 and $T/2$, then the average amount of inventory held per unit time would be $Q_0/2$ as in a single

cycle policy. Unfortunately, this plan is infeasible.

Consider Figure 2.4, which illustrates the amount of echelon inventory in the multiple cycle policy (2,2,3). In addition, the on-hand inventory at the warehouse is represented by the shaded area. At time zero, $Q_0$ is produced at the warehouse and lots of $Q_1$ and $Q_2$ are sent immediately to retailers 1 and 2, respectively. From (2.48), this leaves the warehouse with an on-hand inventory of $Q_2/2$. At time $T/3$, retailer 2 requires another lot of size $Q_2$ from the warehouse. But the on-hand inventory is insufficient to meet this need. To maintain feasibility, the warehouse must produce a lot of size $Q_0$ at time $T/3$.

Recall that if the warehouse produces at times 0 and $T/2$, then the average echelon inventory is $Q_0/2$. In the actual plan, the warehouse must produce at $T/3$ rather than $T/2$ and thus the total echelon inventory for the cycle $T$ is increased by $Q_0(T/2 - T/3) = Q_0T/6$. The increase in the average echelon inventory per unit time is $Q_0/6$. Thus the average warehouse echelon inventory per unit time is

$$Q_0(\tfrac{1}{2} + \tfrac{1}{6}) = \tfrac{2}{3} Q_0. \qquad (2.49)$$

For each facility $i$, including the warehouse, $n_iK_i$ is the cost of set-ups in each cycle of length $T$. Thus, the average set-up cost for facility $i$, per unit time is $n_iK_i/T$.

Given (2.49), we can now express the average cost per unit time of the multiple cycling policy (2,2,3) as

$$\frac{2K_0 + 2K_1 + 3K_2}{T} + h_0(\tfrac{2}{3} Q_0) + h_1(\tfrac{1}{2} Q_1) + h_2(\tfrac{1}{2} Q_2). \qquad (2.50)$$

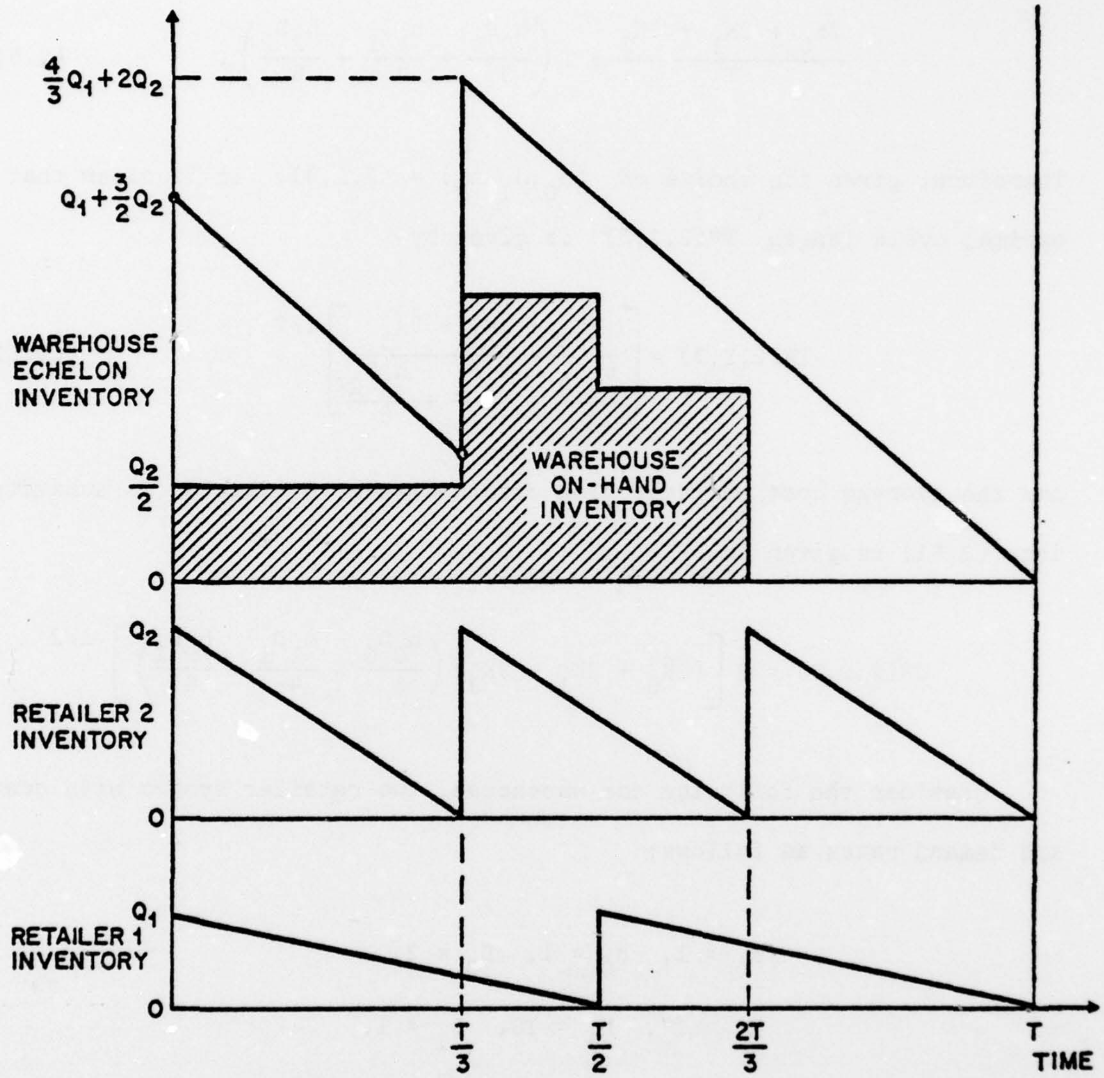



FIGURE 2.4 MULTIPLE CYCLE POLICY WITH $n_0 = 2, n_1 = 2, n_2 = 3$

By stationarity, $Q_i = TD_i/n_i$ for all $i$. Then for this example with $(n_0,n_1,n_2) = (2,2,3)$ we can rewrite (2.50) as

$$\frac{2K_0 + 2K_1 + 3K_2}{T} + T\left(\frac{h_0D_0}{3} + \frac{h_1D_1}{4} + \frac{h_2D_2}{6}\right). \tag{2.51}$$

Therefore, given the choice of $(n_0,n_1,n_2) = (2,2,3)$, it is clear that the optimal cycle length $T^*(2,2,3)$ is given by

$$T^*(2,2,3) = \left[\frac{2K_0 + 2K_1 + 3K_3}{\frac{h_0D_0}{3} + \frac{h_1D_1}{4} + \frac{h_2D_2}{6}}\right]^{1/2} \tag{2.52}$$

and the average cost per unit time obtained when $T^*(2,2,3)$ is substituted into (2.51) is given by

$$C^*(2,2,3) = 2\left[(2K_0 + 2K_1 + 3K_3)\left(\frac{h_0D_0}{3} + \frac{h_1D_1}{4} + \frac{h_2D_2}{6}\right)\right]^{1/2}.$$

Consider the following one warehouse, two retailer system with costs and demand rates as follows:

$$K_0 = 1,\ h_0 = 1,\ D_0 = 2,$$
$$K_1 = 17,\ h_1 = 10,\ D_1 = 1,$$
$$K_2 = 17,\ h_2 = 22,\ \text{and}\ D_2 = 1. \tag{2.54}$$

We can compute the cost of the multiple cycle policy (2,2,3) for this example using (2.53) and we find

$$C^*(2,2,3) = 48.7647.$$

However, the best single cycle policy (1,1) has cost

$$C^*(1,1) = 48.7852$$

and, therefore, we have demonstrated

Theorem 2.11. Among the class of stationary renewal policies, a single cycle policy may not be optimal. □

It is clear, however, that the multiple cycling policy discussed above cannot be the overall optimal solution either. Note that property P1 is violated at time $T/3$. That is, the warehouse produces at time $T/3$ with an on-hand inventory of $Q_2/2$. Clearly, if the warehouse produces $Q_2/2$ fewer units at time zero and $Q_2/2$ more units at time $T/3$, feasibility will be maintained and the average inventory per unit time at the warehouse will be lower than in the multiple cycling policy discussed above. Thus, there is an easily constructed non-stationary policy with lower average cost per unit time than the multiple cycling policy (2,2,3).

Furthermore, it is possible to perform the same type of shifting of production for any general multiple cycle policy, thus demonstrating that a multiple cycle policy cannot be the overall optimal policy. We know by Theorem 2.10, that if the overall optimal policy is stationary then it is a renewal policy. Therefore, we have shown

Theorem 2.12. If the overall optimal policy is stationary, then it is a single cycle policy. □

We would also like to compare the quality of single cycle policies to multiple cycle policies in more general cases. In order to do this, it is necessary to derive a general expression for the echelon holding costs at the warehouse, as we did earlier for the multiple cycle policy (2,2,3). Unfortunately, it is impossible to derive an exact expression based solely on the policy $(n_0,n_1,n_2,\ldots,n_N)$. This can be seen by the following result.

Theorem 2.13. In a multiple cycle policy $(n_0,n_1,\ldots,n_N)$, the echelon holding cost at the warehouse is dependent upon the lot sizes $(Q_1,Q_2,\ldots,Q_N)$.

Proof of Theorem 2.13.

Consider the multiple cycle policy with $(n_0,n_1,n_2) = (2,3,5)$. Assume that the cycle length is $T$. By stationarity, $Q_0 = \frac{3}{2} Q_1 + \frac{5}{2} Q_2$. At time $\frac{T}{3}$, the warehouse has $\frac{1}{2} Q_1 + \frac{1}{2} Q_2$ units on hand and $Q_1$ units are required for retailer 1. If $Q_2 < Q_1$, then the warehouse must produce $Q_0$ units at time $\frac{T}{3}$. If $Q_2 \geq Q_1$, then the warehouse need not produce at time $\frac{T}{3}$ but it will have to produce at $\frac{2T}{5}$. Thus if $Q_2 < Q_1$, the average echelon inventory at the warehouse is $Q_0(\frac{1}{2} + \frac{1}{6})$ and if $Q_1 \leq Q_2$, the average echelon inventory at the warehouse is $Q_0(\frac{1}{2} + \frac{1}{10})$. This demonstrates Theorem 2.13. □

It is clear from the example above, that for $n_0 = 2$, the echelon inventory at the warehouse is determined by which retailer has a simultaneous production point with the warehouse after time zero. Similarly, if $n_0 > 2$, then the echelon inventory at the warehouse is determined by

the sequence of retailers having simultaneous production points with the warehouse. The problem of determining the warehouse echelon inventory for a general sequence of retailers appears impossible but we can analyze a special case.

Theorem 2.14. For the multiple cycle policy $(n_0, n_1, \ldots, n_N)$, if each warehouse production point is a simultaneous production point for the same retailer $i$, then the average echelon inventory at the warehouse, $I$, can be expressed by

$$I = Q_0\left(\frac{1}{2} + \frac{n_0 - 1}{2n_i}\right) \text{ if } n_i \text{ is not an integer multiple of } n_0 \tag{2.55}$$

and is bounded below by

$$I = Q_0\left(\frac{1}{2} + \frac{n_0 - 1}{n_i}\right) \text{ if } n_i \text{ is an integer multiple of } n_0 \text{ greater than } n_0. \tag{2.56}$$

Proof of Theorem 2.14:

If $n_i$ is not an integer multiple of $n_0$, then we can write $n_i = kn_0 + \varepsilon$ where $k$ is an integer and $\varepsilon$ is an integer strictly between $0$ and $n_0$. There are $n_0$ production intervals to be considered. The length of each of these production intervals is either $\frac{k}{n_i} T$ or $\frac{k+1}{n_i} T$, where $T$ is the length of time between the overall renewals.

If there are $b$ intervals of length $\frac{k}{n_i} T$ and $c$ intervals of length $\frac{k+1}{n_i}$, then

$$b\left(\frac{k}{n_i}\right)T + c\left(\frac{k+1}{n_i}\right)T = T \text{ and } b+c = n.$$

It is clear that $b = n_0 - \varepsilon$ and $c = \varepsilon$. Furthermore, the ending echelon inventory for each of the warehouse production intervals takes on the values $\frac{0}{n_i} Q_0, \frac{1}{n_i} Q_0, \ldots, \frac{(\varepsilon-1)}{n_i} Q_0$ for the $\varepsilon$ production intervals with length $\frac{k+1}{n_i} T$ and the values $\frac{\varepsilon}{n_i} Q_0, \frac{(\varepsilon+1)}{n_i} Q_0, \ldots, \frac{(n_0-1)}{n_i} Q_0$ for the $(n_0-\varepsilon)$ production intervals with length $\frac{k}{n_i} T$. This is clear because there are no renewals and thus each value between $\frac{0}{n_i} Q_i$ and $\left(\frac{n_0-1}{n_i}\right) Q_i$ must be taken on. Also, for each production interval that lasts for $\frac{k}{n_1} T$ time units the echelon inventory is $(\frac{\varepsilon}{n_i})Q_0$ units higher at the end than at the start and therefore each $\frac{k}{n_1} T$ type interval must have ending inventory of at least $\frac{\varepsilon}{n_i} Q_0$.

Also, since it requires $T/n_0$ time units to use a lot of size $Q_0$, the amount of inventory used in an interval of length $k/n_i$ is $Q_0(kn_0/n_i)$. Similarly, the amount of inventory used in an interval of length $(k+1)/n_i$ is $Q_0((k+1)n_0/n_i)$. Therefore we can write the total warehouse echelon inventory for a cycle as

$$\sum_{\substack{\text{All production}\\ \text{intervals}}} (\text{Average height of production interval}) \cdot (\text{Length of production interval})$$

$$= \sum_{\substack{\text{All production}\\ \text{intervals with}\\ \text{length } Tk/n_i}} \left(\frac{\text{Inventory used in production interval}}{2} + \text{Ending inventory}\right) \cdot (Tk/n_i)$$

$$+ \sum_{\substack{\text{All production}\\ \text{intervals with}\\ \text{length } T(k+1)/n_i}} \left(\frac{\text{Inventory used in production interval}}{2} + \text{Ending inventory}\right) \cdot (T(k+1)/n_i)$$

$$= (n_0-\varepsilon)\cdot\left(\frac{Q_0kn_0}{n_i}\right)\left(\frac{Tk}{2n_i}\right) + \left(\frac{Q_0\varepsilon}{n_i} + \frac{Q_0(\varepsilon+1)}{n_i} + \ldots + \frac{Q_0(n_0-1)}{n_i}\right)\cdot\left(\frac{Tk}{n_i}\right)$$

$$+ \varepsilon\left(\frac{Q_0(k+1)n_0}{n_i}\right)\left(\frac{T(k+1)}{2n_i}\right) + \left(Q_0\left(\frac{0}{n_i}\right) + Q_0\left(\frac{1}{n_i}\right) + \ldots + Q_0\left(\frac{(\varepsilon-1)}{n_i}\right)\right)\cdot\left(\frac{T(k+1)}{n_i}\right)$$

$$= (n_0-\varepsilon)\left(\frac{Tk}{n_i}\right)Q_0\left\{\frac{n_0k+n_0+\varepsilon-1}{2n_i}\right\} + \varepsilon\left(\frac{T(k+1)}{n_i}\right)Q_0\left\{\frac{(\varepsilon-1)+(k+1)n_0}{2n_i}\right\},$$

$$= (n_0-\varepsilon)\left(\frac{Tk}{n_i}\right)Q_0\left\{\frac{n_i+n_0-1}{2n_i}\right\} + \varepsilon\left(\frac{T(k+1)}{n_i}\right)Q_0\left\{\frac{n_i+n_0-1}{2n_i}\right\},$$

since $n_i = kn_0+\varepsilon$ and

$$= \left\{\frac{n_i+n_0-1}{2n_i}\right\}TQ_0.$$

**Therefore, the average echelon inventory held at the warehouse per unit time can be expressed as**

$$\left\{\frac{1}{2} + \frac{n_0-1}{2n_i}\right\}Q_0.$$

If, on the other hand, $n_i = kn_0$ where $k$ is an integer and greater than 1, then the first warehouse production after time zero occurs no later than at time $T(\frac{k-1}{n_i})$. This warehouse production cannot occur at time $\frac{T}{n_0}$ or the system would have a renewal at that point in time. The ending echelon inventory for this warehouse production interval is at least as large as $Q_0(1 - \frac{k-1}{n_i} T \Big/ \frac{T}{n_0}) = Q_0(\frac{1}{k})$. In fact, because $n_i$ is a multiple of $n_0$, the ending echelon inventory cannot be any lower than $Q_0(\frac{1}{k})$ until the final warehouse production interval in the cycle. If the warehouse production intervals have lengths $\frac{k-1}{n_1}, \underbrace{\frac{k}{n_1}, \frac{k}{n_1}, \ldots, \frac{k}{n_1}}_{n_0-2}, \frac{k+1}{n_1}$, then the ending echelon inventory for all the warehouse production intervals, except the last, is exactly $Q_0(\frac{1}{k})$. So, clearly, the average echelon inventory for this plan is a lower bound for the average echelon inventory in the actual production plan.

If we compute the average inventory, $I$, for this plan we find

$$I = Q_0\left\{\underbrace{\left(\frac{1}{k}\,\frac{(n_1-k-1)}{n_1}\right)}_{\substack{\text{Ending echelon}\\ \text{inventory}}} + \underbrace{((\frac{k-1}{n_1})(\frac{k-1}{k})\frac{1}{2}) + (\frac{n_0-2}{n_2})(\frac{k}{n_1}) + (\frac{k+1}{n_1}(\frac{k+1}{k})\frac{1}{2})}_{\substack{\text{Average inventory used for}\\ \text{each warehouse production interval}}}\right\}$$

$$= Q_0\left\{\frac{1}{2} + \frac{n_0-1}{n_1}\right\}. \quad \square$$

There do exist some special cases of multiple cycle policies where all warehouse production points must be simultaneous production points for a particular retailer.

Corollary 2.15. The formula (2.55) may be used to exactly express the average echelon inventory at the warehouse for the multiple cycle policy $(n_0, n_1, \ldots, n_N)$ if

(a) $n_i \neq n_0$, for some $i$, and $n_j = n_0$ for all $j \neq i$ or

(b) $n_i = n_j$ for all $i,j$ not equal to zero.

Proof of Corollary 2.15.

It is only necessary to show that Theorem 2.14, for $n_i$ not a multiple of $n_0$, may be applied. In both cases (a) and (b), all warehouse productions are triggered by a retailer $i$ production. In either case, if $n_i = kn_0$, then all the $n_i$ are divisible by $n_0$ and the multiple cycle policy is equivalent to the single cycle policy $(1, \frac{n_1}{n_0}, \frac{n_2}{n_0}, \ldots, \frac{n_N}{n_0})$. □

Suppose we evaluate the formulas (2.55) and (2.56) for each of the retailers in the multiple cycle policy $(n_0, n_1, \ldots, n_N)$. Denote the minimum of these quantities by $L(n_0, n_1, \ldots, n_N)$. For some special cases of multiple cycle policies $L(n_0, n_1, \ldots, n_N)$ is a lower bound on the average echelon inventory held at the warehouse.

Corollary 2.16. $L(n_0, n_1, \ldots, n_N)$ is a lower bound on the average echelon inventory at the warehouse if

(a) $n_0 = 2$ or

(b) $n_i > n_0 n_j$ for $j \neq i$.

Proof of Corollary 2.16.

(a) If $n_0 = 2$, both warehouse production points are simultaneous production points for a particular retailer. Without additional information concerning the lot sizes, it is impossible to determine which retailer

should be used to evaluate (2.55) or (2.56). By evaluating (2.55) or (2.56) for all the retailers and choosing the minimum, we must have a lower bound on the actual average echelon inventory.

(b) If $n_i > n_0 n_j$ then $\frac{1}{n_i} < \frac{1}{n_0 n_j}$ and clearly $L(n_0, n_1, \ldots, n_N)$ is found by evaluating (2.55) or (2.56) for $n_i$. Suppose the $(\ell+1)$st warehouse production is triggered by retailer $j$ at time $\frac{k_j}{n_j} T$ where $k_j$ is an integer. Then the total echelon inventory is $Q_0(\frac{\ell}{n_0} T - \frac{k_j}{n_j} T)$ higher for this plan than if the warehouse produced exactly at time $\frac{\ell}{n_0} T$. Since $\ell$, $n_0$, $k_j$ and $n_j$ are all integers, $T(\frac{\ell}{n_0} - \frac{k_j}{n_j}) > \frac{T}{n_0 n_j}$. Since $\frac{1}{n_i} < \frac{1}{n_0 n_j}$, by assumption, there is a retailer $i$ production between $\frac{k_j}{n_j} T$ and $\frac{\ell T}{n_0}$ and the total echelon inventory would be lower if it were possible to delay the $(\ell+1)$st warehouse production point and make it a simultaneous production point for retailer $i$.

This same argument will hold for all warehouse production points and therefore the possibly infeasible plan where all warehouse production points are simultaneous production points for retailer $i$ has average echelon inventory no larger than the actual plan. Since $n_i$ minimizes $L(n_0, n_1, \ldots, n_N)$, this is clearly a lower bound on the average echelon inventory. □

In general, it is possible to compute a lower bound on the average echelon inventory for the multiple cycle policy $(n_0, n_1, \ldots, n_N)$ that is better than the obvious lower bound of $Q_0/2$. Simply determine the retailer productions closest but prior to $\frac{T}{n_0}$, $\frac{2T}{n_0}$, ..., and $\frac{(n_0-1)T}{n_0}$, and then compute the average echelon inventory as if these were the warehouse production points.

This bound can be achieved in certain instances. Consider the multiple cycle policy with $(n_0,n_1,n_2) = (3,4,5)$. The warehouse produces at either $\frac{T}{5}$ or $\frac{T}{4}$. If $Q_1 \geq Q_2$, the warehouse produces at $\frac{T}{4}$. Also, the warehouse produces at either $\frac{T}{2}$ or $T(\frac{3}{5})$. If $Q_2 \geq Q_1$, the warehouse produces at $T(\frac{3}{5})$. Thus if $Q_2 = Q_1$, the warehouse produces at $\frac{T}{4}$ and $T(\frac{3}{5})$ and the lower bound described above is achieved.

If $Q_1 = Q_2$ in the multiple cycle policy (3,4,5) then the average warehouse echelon inventory is $Q_0(\frac{1}{2} + \frac{3}{20})$. However, for this case $L(3,4,5) = Q_0 \cdot \min(\frac{1}{2} + \frac{1}{5}, \frac{1}{2} + \frac{1}{4}) = Q_0(\frac{1}{2} + \frac{1}{5})$. Thus $L(n_0,n_1,\ldots,n_N)$ may not be a lower bound for general multiple cycle policies although it is an easily computed approximation.

The above results give us a way to express the average warehouse echelon inventory, or a lower bound on that quantity, for multiple cycle policies. Given such an expression, we can write out a lower bound on the average cost per unit time for a multiple cycle policy solely as a function of the policy $(n_0,n_1,\ldots,n_N)$. We can then compute the best $T^*(n_0,n_1,\ldots,n_N)$ and the cost $C^*(n_0,n_1,\ldots,n_N)$ as was done earlier for the multiple cycle policy (2,2,3) in equations (2.52) and (2.53). We can then compute the best unrestricted $n_i$'s and perform the same iterative procedure as was discussed earlier for the determination of single cycle policies. The only difference is that we must ensure $n_i \geq n_0$ for all i.

Using this iterative procedure, 50 of the randomly generated problems used to test the heuristic procedures were evaluated to find multiple cycle policies with lower average cost per unit time than the best single cycle policy. In all 50 problems, no such multiple cycle policy was found.

There may indeed be such policies but none were found with this heuristic search technique.

## 2.13 Summary

There appear to be few cases where one can guarantee the overall optimality of single cycle policies. We have demonstrated that single cycle policies need not be optimal among the class of:

(1) all policies;

(2) separate retailing and single cycle policies;

(3) stationary policies;

(4) stationary policies satisfying properties P1-P5;

and (5) stationary renewal policies.

However, single cycle policies are optimal for the classes of problems where:

(1) all retailers are identical;

(2) all retailers have identical ratios;

and (3) the overall optimal policy is stationary.

In many of the proofs presented, non-stationary policies were constructed with lower cost than single cycle policies. But, it is precisely the stationarity of single cycle policies that makes them appealing when one considers the practical concerns of implementation. The single cycle policy is not only stationary in lot size but in time as well. Each facility not only produces the same lot size each time it produces but the time between these productions is constant as well. This property is not present in multiple cycle policies. Also, the presence of renewal points in cycling policies offer the system opportunities

to restart the system with a revised policy should any of the system parameters change. Furthermore, the optimal single cycle policy or near optimal single cycle policy is easily computed. Graves and Schwarz (1977) have presented a method for determining good single cycle policies in general arborescent production inventory by computing single cycle policies for many one warehouse, N retailer systems. In a procedure such as this, the efficient computation of good single cycle policies is crucial.

The empirical evidence also suggests that although one cannot guarantee the optimality of single cycle policies, they do appear to be close to the optimum, at least for small problems. Further research is necessary to establish exact bounds on the difference in costs between the best single cycle solution and the true optimum for any general N retailer problem.

# CHAPTER 3

# DYNAMIC ONE WAREHOUSE, N RETAILER PROBLEMS

## 3.1 Introduction

In this chapter we examine the dynamic one warehouse, N retailer production inventory system. We begin by introducing the basic assumptions and notation. We review the Wagner-Whitin and Zangwill algorithms for the single facility and one warehouse, one retailer problems, respectively. We show how to represent the one warehouse, N retailer problem as a single source, concave cost, network flow model. A dynamic programming algorithm is presented for this case. Easily computed upper and lower bounds for the optimal solution value are developed and several special cases are discussed.

## 3.2 Basic Model

The basic structure of facilities is identical to that discussed in Chapter 2. The warehouse will be referred to as facility 0 and the N retailers will again be referred to as facilities 1 through N, respectively.

In the dynamic demand case, the demand at retailer i is assumed to occur at a known rate $r_t^i$ in period t, where $t = 1,2,\ldots,T$. The demand rate may differ from one facility to another and from one time period to another. We will assume that the warehouse experiences no external demand.

Inventory is maintained between stages and the output of the warehouse in period t can be used by any of the retailers in the same period. Time lags can be included by redefining the decision variables to account for

the lead time. Also, we assume that backorders are not allowed. At each facility, there is a production cost and a holding cost which may differ from one period to the next. In particular, let

$x_t^j$ = Production at facility $j$ in period $t$,

$I_t^j$ = On-hand inventory at facility $j$ at the end of period $t$,

$P_t^j(x_t^j)$ = Production cost for facility $j$ in period $t$,

$H_t^j(I_t^j)$ = Holding cost for facility $j$ in period $t$, where

$j = 0,1,\ldots,N$ and $t = 0,1,2,\ldots,T$.

In order to be consistent with the previous research on related problems, we charge the holding cost at the warehouse against the on-hand inventory and not the echelon inventory as in the continuous demand case. We will also assume that the production costs and holding costs, $P_t^j(x_t^j)$ and $H_t^j(I_t^j)$ are concave and non-decreasing for non-negative arguments. Without loss of generality, we can also assume that $I_0^j = I_T^j = 0$ for all $j$.

The objective function is the sum of the holding and set-up costs. The constraints reflect the inventory balance and the no backlogging assumption. We can therefore express the problem of determining the best sequence of production and inventory quantities by:

$$\text{Minimize} \quad z = \sum_{t=1}^{T} \sum_{j=0}^{N} [P_t^j(x_t^j) + H_t^j(I_t^j)] \tag{3.1}$$

$$\text{subject to} \quad I_t^0 = I_{t-1}^0 + x_t^0 - \sum_{j=1}^{N} x_t^j, \quad t = 1,\ldots,T \tag{3.2}$$

$$I_t^j = I_{t-1}^j + x_t^j - r_t^j, \quad j = 1,\ldots,N \quad t = 1,\ldots,T \tag{3.3}$$

$$x_t^j \geq 0, \quad I_t^j \geq 0 \tag{3.4}$$

We will derive a dynamic programming algorithm for this problem which is a generalization of Zangwill's (1963) algorithm for the serial production inventory system. In addition, we will develop upper and lower bounds on the optimal solution value for this problem which made use of Wagner and Whitin's (1958) solution technique for the single facility problem. Before discussing the algorithm for the one warehouse, N retailer system, it is therefore necessary to review these two previously examined cases.

## 3.3 A Dynamic Programming Algorithm for the Single Facility Problem

In this section, we will review the Wagner-Whitin (1958) solution technique for the single facility problem. The algorithm exploits the structure of the problem when determining an optimal solution.

For the single facility problem, we may rewrite the mathematical programming formulation, (3.1)-(3.4), as

$$\text{Minimize} \quad z = \sum_{t=1}^{T} [P_t^1(x_t^1) + H_t^1(I_t^1)] \tag{3.5}$$

$$\text{subject to} \quad I_t^1 = I_{t-1}^1 + x_t^1 - r_t^1, \quad t = 1,\ldots,T \tag{3.6}$$

$$x_t^1 \geq 0, \quad I_t^1 \geq 0, \quad t = 1,\ldots,T. \tag{3.7}$$

Since the objective function is concave, the solution to the above problem must occur at an extreme point of the set of feasible solutions. In this case,

$$x_t^1 \cdot I_{t-1}^1 = 0 \quad \text{for all } t \text{ in an optimal solution.} \tag{3.8}$$

Therefore, $x_t^1 \in \{0, r_t^1, r_t^1+r_{t+1}^1, \ldots, r_t^1+r_{t+1}^1+\ldots+r_T\}$. The dynamic programming algorithm to be developed will take advantage of this form for an optimal solution.

Let $f_k(y)$ = minimum cost for the first $k$ periods when the on-hand inventory at the end of period $k$ is $y$.

Then

$$f_k(y) = \min_{x_k^1}\{P_k^1(x_k^1) + H_k^1(y) + f_{k-1}(y - x_k^1 + r_k^1)\}. \tag{3.9}$$

Since $x_k^1 \cdot I_{k-1}^1 = 0$,

$$f_k(y) = \min\left\{\begin{matrix} P_k^1(r_k^1+y) + H_k^1(y) + f_{k-1}(0) \\ P_k^1(0) + H_k^1(y) + f_{k-1}(y+r_k^1) \end{matrix}\right\}. \tag{3.10}$$

Furthermore,

$$f_k(y) = \min_i Y_k(i) \quad \text{where} \tag{3.11}$$

$$Y_k(i) = \sum_{j=i+1}^{k} \left\{ P_j^1(0) + H_j^1(y + \sum_{u=j+1}^{k} r_u^1) \right\}$$

$$+ P_i^1(y + \sum_{u=i}^{k} r_u^1) + H_i^1(y + \sum_{u=i+1}^{k} r_u) \tag{3.12}$$

$$+ f_{i-1}(0).$$

Thus $Y_k(i)$ = the cost of periods 1 through $k$ under the assumptions that an optimal policy is followed for the first $i-1$ periods and the inventory at the end of period $i-1$ is zero, and an order arrives at the beginning of period $i$ satisfying the demand for periods $i$ through $k$ with the on-hand inventory at the end of period $k$ equalling $y$.

We need not compute $f_k$ other than for $y = 0$. Thus the task of finding the optimal solution requires less effort than applying a standard backward recursion dynamic programming algorithm. However, for the one warehouse, one retailer problem the backward recursion Zangwill algorithm (1969) will be developed in the next section.

## 3.4 A Dynamic Programming Algorithm for the One Warehouse, One Retailer Problem

In this section, we review a dynamic programming algorithm for the one warehouse, one retailer production inventory system. Zangwill (1969)

has presented a dynamic programming algorithm for the serial deterministic demand production inventory system. Clearly, if the serial system has only two stages, then this system is identical to the one warehouse, one retailer problem described in the preceding section. Zangwill has shown that the serial system may be represented by a single source network flow model. To illustrate this, consider Figure 3.1 which depicts the single source network where T = 4 and node (j,t) represents facility j and period t.

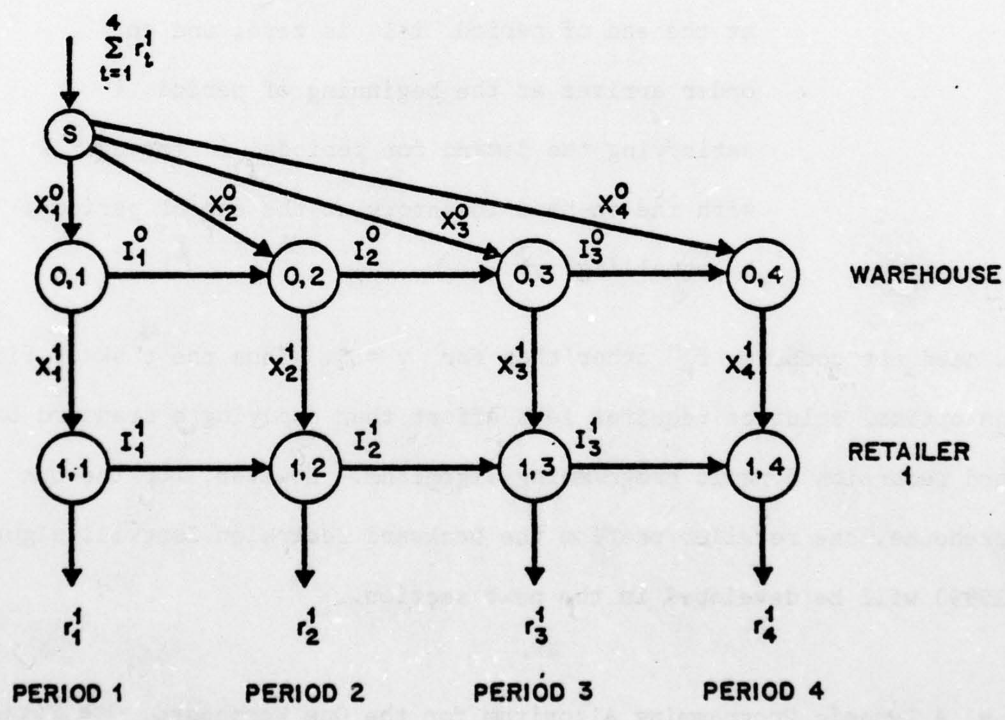



FIGURE 3.1 NETWORK REPRESENTATION OF ONE WAREHOUSE, ONE RETAILER SYSTEM

Clearly, the conservation of flows in the network in Figure 3.1 ensures that the constraints (3.2) and (3.3) are satisfied. By charging the appropriate cost for the flow along each of the arcs, it is clear that finding the minimum cost solution to the mathematical programming problem (3.1)-(3.4) with $N = 1$, is equivalent to finding the minimum cost flow in the network representation.

Since the cost functions are concave on the arc flows and the network has a single source, we know that at least one "extreme flow" is an optimal solution (Zangwill, 1968). That is, at most one arc entering a node will have a positive flow.

Therefore, the flow into each node, $f$, can be expressed by

$$f = \sum_{i=a}^{b} r_i^1 \quad \text{where} \quad t \le a \le b \le T. \tag{3.13}$$

In this case, it is said that the extreme flow has the "exact requirements" property. Clearly, for any node $(1,t)$, $a = t$ in (3.13) to prevent backorders. Thus,

$$I_t^1 = \begin{cases} \sum_{i=t+1}^{b} r_i^1, & t < b \le T, \\ 0 \quad , & b = t. \end{cases} \tag{3.14}$$

From (3.13), if the flow into $(0,t)$ is $\sum_{i=a}^{b} r_i^1$, $t \le a \le b \le T$, then there are two cases to be considered: $a = t$ and $a > t$. If $a = t$, then $x_t^1 > 0$ to insure that $r_t^1$ is satisfied. Thus, if $a = t$, there exists an integer $c$, $t \le c \le b$ such that

$$x_t^1 = \sum_{i=a}^{c} r_i^1, \text{ and } I_t^0 = \sum_{i=c+1}^{b} r_i^1. \tag{3.15}$$

If $a > t$, then $x_t^1 = 0$ or the two arcs into node $(1,t)$ would have positive flow and therefore

$$I_t^0 = \sum_{i=a}^{b} r_i^1. \tag{3.16}$$

Define $K_t^j(a,b)$ to be the optimal cost of completing the production schedule if $\sum_{i=a}^{b} r_i^1$ units enter node $(j,t)$. Also let

$$P_t^j(a,b) = P_t^j\left(\sum_{i=a}^{b} r_i^1\right) \text{ and}$$

$$H_t^j(a,b) = H_t^j\left(\sum_{q=a}^{b} r_i^1\right).$$

If $a = t$ in (3.13), for the retailer node $(1,t)$, then by (3.14),

$$\begin{aligned} K_t^1(t,b) &= H_t^1(t+1,b) + K_{t+1}^1(t+1,b) \\ &= \sum_{i=t}^{b} H_i^1(i+1,b), \end{aligned} \tag{3.17}$$

which holds for $1 \leq t \leq T-1$, $t \leq b \leq T$; if $t = T$ then $K_T^1(T,T) = 0$.

If $a = t$ in (3.13), for the warehouse node $(0,t)$, then by (3.15),

$$K_t^0(t,b) = \min_{t \leq c \leq b} \{P_t^1(t,c) + H_t^0(c+1,b) + K_t^1(t,c) + K_{t+1}^0(c+1,b)\}, \tag{3.18}$$

for $1 \leq t \leq T-1$ and $t \leq b \leq T$; if $t = T$, $K_T^0(T,T) = P_T^1(T,T)$.

If $a > t$ in (3.13) for the warehouse node $(0,t)$, then by (3.16),

$$K_t^0(a,b) = H_t^0(a,b) + K_{t+1}^0(a,b), \tag{3.19}$$

for $2 \leq t \leq T-1$ and $t < a \leq b \leq T$. If $t = 1$, then $a = 1$ and $K_1^0(1,b)$ from (3.18) can be applied.

To complete the algorithm, we must determine the decisions at the source node, i.e. how much to produce at the warehouse in each period. Let $K_t^s(a,T)$ = minimum cost of shipping $\sum_{t=a}^{T} r_t^1$ units from the source to satisfy the requirements at nodes $(1,a),(1,a+1),\ldots,(1,T)$, that is, the optimal cost of satisfying demands $r_a^1,\ldots,r_T^1$ through production in periods $t,t+1,\ldots,T$. Demand is assumed satisfied for the first $a-1$ periods from production in the periods prior to $t$. Also if $a = t$, we must produce in period $t$ at the warehouse.

Therefore,

$$K_t^s(a,T) = \underset{\max(t,a-1)\leq c\leq T}{\text{minimum}} \{P_t^0(a,c) + K_t^0(a,c) + K_{t+1}^s(c+1,T)\}, \tag{3.20}$$

where $2 \leq t \leq T-1$ and $t \leq a \leq T$. If $c = a-1$, $x_t^0 = 0$ by convention and if $t = 1$, we use (3.20) only with $a = 1$.

We begin the algorithm with $K_T^1(T,T) = 0$ and work backward until $K_1^s(1,T)$ is evaluated. $K_1^s(1,T)$ is the minimum cost attainable in this network.

Zangwill (1966,1969) has extended the single facility model discussed in Section 3.3 and the one warehouse, one retailer model discussed above to include the possibility of backlogging of external

demand. Each retailer node (1,T) in Figure 3.1 is revised to include additional arcs to represent the flow back in time (Figure 3.2). If we let

$I_t^{1+}$ = inventory on-hand at the end of period $t$ at retailer 1,

$I_t^{1-}$ = inventory short at the end of period $t$ at retailer 1,

$H_t^{1+}(I_t^{1+})$ = inventory carrying cost associated with $I_t^{1+}$,

and $H_t^{1-}(I_t^{1-})$ = shortage cost associated with $I_t^{1-}$.

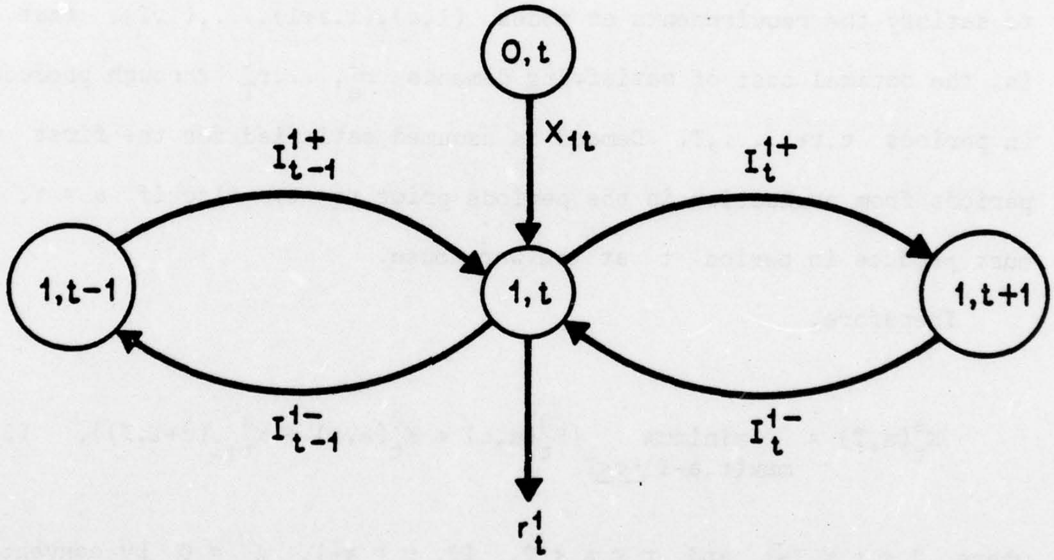



**FIGURE 3.2 NODE (1,t) WITH BACKLOGGING**

We will omit the development of the recursion because we will only extend the no backlogging case to the one warehouse, N retailer system. An extension to include backlogging is possible. But as we shall see in the next section, the one warehouse, N retailer algorithm is quite

complicated computationally and such extensions will only further exacerbate this problem.

## 3.5 A Dynamic Programming Algorithm for the One Warehouse, N Retailer Problem

It is possible to generalize the one warehouse, one retailer dynamic programming algorithm presented in Section 3.4 to solve the one warehouse, $N$ retailer system. Veinott (1969) first noted that this generalization was possible but he failed to point out the basic differences between the generalized algorithm and the Zangwill procedure. Veinott also presented an efficient algorithm for the general arborescent problem under a severe set of restrictions on the cost structure that is similar to Love's nested algorithm (1972) for facilities in series.

We begin by representing the one warehouse, $N$ retailer system by a single source network flow model. To illustrate this, consider Figure 3.3 which depicts the single source network for two retailers where $T = 4$ and node $(j,t)$ represents facility $j$ and period $t$.

Clearly, the conservation of flows in the network in Figure 3.3 ensures that the constraints (3.2) and (3.3) are satisfied. By charging the appropriate cost for the flow along each of the arcs, it is clear that finding the minimum cost solution to the mathematical programming problem (3.1)-(3.4) is equivalent to finding the minimum cost flow in the network representation.

Since the cost functions are concave on the arc flows and the network has a single source, we know that at least one extreme flow is an optimal solution. That is, at most one arc entering a node will have a positive flow.

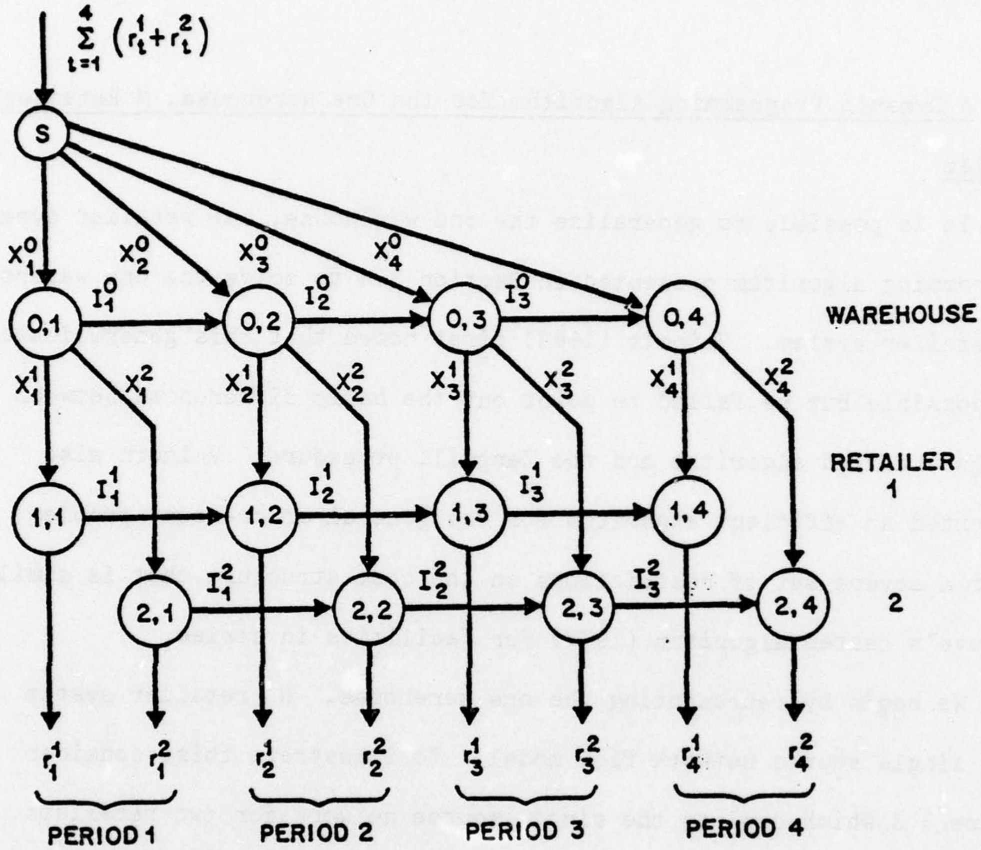



FIGURE 3.3 NETWORK REPRESENTATION OF ONE WAREHOUSE, N RETAILER SYSTEM

In order to clarify the exposition for the dynamic programming algorithm, we will discuss only the two retailer case but the results can easily be generalized.

As in the one retailer case, the flow, f, into a retailer node (j,t), i.e. j > 0, can be expressed by

$$f = \sum_{i=t}^{b} r_i^j \quad \text{where} \quad t < b \le T,$$

and therefore

$$I_t^j = \begin{cases} \sum_{i=t+1}^{b} r_i^j, & t < b \le T, \\ 0 \quad, & b = t. \end{cases} \tag{3.21}$$

Define $K_t^j(t,b)$, as before, to be the optimal cost of shipping $\sum_{i=t}^{b} r_i^j$ units from node $(j,t)$ to their respective destinations. Then, it is clear that

$$K_t^j(t,b) = H_t^j(t+1,b) + K_{t+1}^j(t+1,b)\ , \tag{3.22}$$

which holds for $j > 0$, $1 \le t \le T-1$, $t \le b \le T$; if $t = T$, then $K_T^j(T,T) = 0$ for $j > 0$. Also,

$$K_t^j(t,t) = H_t^j(t+1,t) + K_{t+1}^j(t+1,t), \tag{3.23}$$

which by convention,

$$= 0 + 0.$$

The recursion is somewhat more complicated for the warehouse nodes $(0,t)$. The extreme flow property demonstrates that the flow, $f$, into a node $(0,t)$ can be expressed by

$$f = \sum_{i=a}^{b} r_i^1 + \sum_{i=c}^{d} r_i^2, \tag{3.24}$$

where there are 3 cases for the values of a, b, c and d. Either

(i) $t \leq a \leq b \leq T$ and $t \leq c \leq d \leq T$,

or (ii) $a = b+1$, $b \leq T$ and $t \leq c \leq d \leq T$,

or (iii) $t \leq a \leq b \leq T$ and $c = d+1$, $d \leq T$.

Case (i) is exactly analogous to the flow into a warehouse node for the one retailer case. By convention, $\sum_{i=b+1}^{b} r_i^j = 0$, for $j = 1,2$, and case (ii) and (iii) are necessary because the flow into a warehouse node may only consist of the sum of requirements for a single retailer.

Let $K_t^0(a,b,c,d)$ be the optimal cost of shipping $\sum_{i=a}^{b} r_i^1 + \sum_{i=c}^{d} r_i^2$ units to their respective destinations. Also define

$$H_t^0(a,b,c,d) = H_t^0\left(\sum_{i=a}^{b} r_i^1 + \sum_{i=c}^{d} r_i^2\right) \text{ and}$$

$$P_t^0(a,b,c,d) = P_t^0\left(\sum_{i=a}^{b} r_i^1 + \sum_{i=c}^{d} r_i^2\right).$$

By convention,

$$H_t^0(b+1,b,d+1,d) = P_t^0(b+1,b,d+1,d) = 0.$$

We begin by considering case (i) $t \leq a \leq b \leq T$, $t \leq c \leq d \leq T$ for which there are four subcases, since both a and c either equal t or are greater than t.

(i-1) If $a = t$ and $c = t$, there exist integers f and g, $t \leq f \leq b$, $t \leq g \leq d$ such that

$$x_t^1 = \sum_{i=a}^{f} r_i^1, \quad x_t^2 = \sum_{i=c}^{g} r_i^2 \text{ and } I_t^0 = \sum_{i=f+1}^{b} r_i^1 + \sum_{i=g+1}^{d} r_i^2. \qquad (3.25)$$

(i-2) If $a = t$ and $c > t$, there exists an integer $f$, $t \le f \le b$, such that

$$x_t^1 = \sum_{i=a}^{f} r_i^1, \quad x_t^2 = 0 \quad \text{and} \quad I_t^0 = \sum_{i=f+1}^{b} r_i^1 + \sum_{i=c}^{d} r_i^2. \tag{3.26}$$

(i-3) If $a > t$ and $c = t$, there exists an integer $g$, $t \le g \le d$, such that

$$x_t^1 = 0, \quad x_t^2 = \sum_{i=c}^{g} r_i^2 \quad \text{and} \quad I_t^0 = \sum_{i=a}^{b} r_i^1 + \sum_{i=g+1}^{d} r_i^2. \tag{3.27}$$

(i-4) If $a > t$ and $c > t$, then

$$x_t^1 = 0, \quad x_t^2 = 0 \quad \text{and} \quad I_t^0 = \sum_{i=a}^{b} r_i^1 + \sum_{i=c}^{d} r_i^2. \tag{3.28}$$

We can now determine the recursive equations for each of these 4 subcases of case (i).

(i-1) If $a = t$ and $c = t$, by (3.25),

$$K_t^0(t,b,t,d) = \min_{\substack{t \le f \le b \\ t \le g \le d}} \left\{ \begin{array}{l} P_t^1(t,f) + K_t^1(t,f) + H_t^0(f+1,b,g+1,d) \\ \quad + P_t^2(t,g) + K_t^2(t,g) + K_{t+1}^0(f+1,b,g+1,d) \end{array} \right\}; \tag{3.29}$$

for $1 \le t \le T-1$;

if $t = T$, $K_T^0(T,T,T,T) = P_T^1(T,T) + P_T^2(T,T)$.

(i-2) If $a = t$ and $c > t$, by (3.26),

$$K_t^0(t,b,c,d) = \min_{t \le f \le b} \left\{ \begin{array}{l} P_t^1(t,f) + K_t^1(t,f) + H_t^0(f+1,b,c,d) \\ \quad + K_{t+1}^0(f+1,b,c,d) \end{array} \right\}, \tag{3.30}$$

for $1 \le t \le T-1$.

(i-3) If $a > t$ and $c = t$, by (3.27),

$$K_t^0(a,b,t,d) = \min_{t \le g \le d} \left\{ \begin{array}{l} P_t^2(t,g) + K_t^2(t,g) + H_t^0(a,b,g+1,d) \\ \quad + K_{t+1}^0(a,b,g+1,d) \end{array} \right\}, \tag{3.31}$$

for $1 \le t \le T-1$.

(i-4) If $a > t$ and $c > t$, by (3.28),

$$K_t^0(a,b,c,d) = H_t^0(a,b,c,d) + K_{t+1}^0(a,b,c,d). \tag{3.32}$$

Thus far the $N$ retailer case is a straightforward extension of Zangwill's one retailer algorithm. But, the development of recursion equations for cases (ii) and (iii) is somewhat more complicated. Let us examine case (ii) more carefully.

Case (ii) corresponds to $\sum_{i=b+1}^{b} r_i^1 = 0$ units for retailer 1 and $\sum_{i=c}^{d} r_i^2$ units for retailer 2 carried into the warehouse node $(0,t)$. There is only one possible way to arrive at such a situation. In either the recursion (3.29) or (3.30), for some preceding time period, $f$ was chosen equal to $b$. That is, the requirements up to and including period $b$ were produced and then held at retailer 1. Clearly, the retailer 1 requirements for period $b+1$ and after have not yet been produced at the warehouse.

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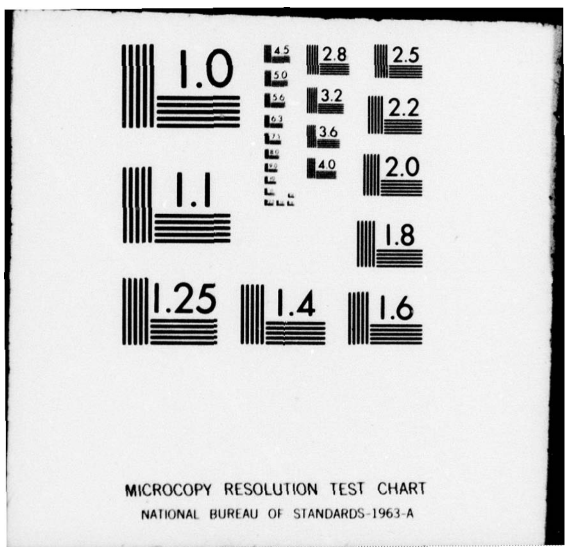
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Let us suppose now that $b = t-1$. Then in case (ii), $\sum_{i=c}^{d} r_i^2$ units flow into node $(0,t)$ but as discussed above the requirements for retailer 1 for period $t$ and after have not yet been produced. Clearly, to maintain feasibility at least $r_t^1$ units must be produced at the warehouse in period $t$. This implies that the two arcs into node $(0,t)$ both have positive flow which is impossible by the property of extreme flows. A similar argument for case (iii) illustrates that $d = t-1$ yields the same contradiction. To ensure that the dynamic programming algorithm only examines extreme flows we define

$$\begin{aligned} K_t^0(t,t-1,c,d) &= \infty \quad \text{for} \quad c \leq d \quad \text{and} \\ K_t^0(a,b,t,t-1) &= \infty \quad \text{for} \quad a \leq b. \end{aligned} \tag{3.33}$$

Given these definitions, we may now write the recursive equations for cases (ii) and (iii). For case (ii) there are two subcases since either $c = t$ or $c > t$. If $c = t$, $x_t^2 > 0$ and we may write

$$K_t^0(b+1,b,t,d) = \min_{t \leq g \leq d} \left\{ \begin{aligned} &P_t^2(t,g) + K_t^2(t,g) + K_{t+1}^0(b+1,b,g+1,d) \\ &+ H_t^0(b+1,b,t,g) \end{aligned} \right\}. \tag{3.34}$$

If $c > t$, $x_t^2 = 0$ and we may write

$$K_t^0(b+1,b,c,d) = H_t^0(b+1,b,c,d) + K_{t+1}^0(b+1,b,c,d). \tag{3.35}$$

Similarly, for case (iii) the two subcases are whether $a = t$ or $a > t$. If $a = t$, $x_t^1 > 0$ and we may write

$$K_t^0(t,b,d+1,d) = \min_{t\leq f\leq b} \left\{ \begin{array}{l} P_t^1(t,f) + K_t^1(t,f) + K_{t+1}^0(f+1,b,d+1,d) \\ + H_t^0(t,f,d+1,d) \end{array} \right\}. \quad (3.36)$$

If $a > t$, $x_t^1 = 0$ and we may write

$$K_t^0(a,b,d+1,d) = H_t^0(a,b,d+1,d) + K_{t+1}^0(a,b,d+1,d). \quad (3.37)$$

Finally,

$$K_t^0(b+1,b,d+1,d) = 0. \quad (3.38)$$

To complete the algorithm, we must determine the decisions at the source node, $s$, i.e. how much to produce at the warehouse in each period. Let $K_t^s(a,T,c,T)$ = minimum cost of shipping $\sum_{t=a}^{T} r_t^1 + \sum_{t=c}^{T} r_c^2$ units from the source to satisfy the requirements at $(1,a),(1,a+1),\ldots,(1,T),(2,c),(2,c+1),\ldots,(2,T)$, that is, the optimal cost of satisfying demands $r_a^1,\ldots,r_T^1,r_c^2,\ldots,r_T^2$ through productions in periods $t,t+1,\ldots,T$. Demand for each retailer is assumed satisfied from production in the periods prior to $t$. Also if $a = t$ or $c = t$, we must produce in $t$ at the warehouse.

Therefore

$$K_t^s(a,T,c,T) = \min_{\substack{\max(t,a-1)\leq f\leq T \\ \max(t,c-1)\leq g\leq T}} \left\{ \begin{array}{l} P_t^0(a,f,c,g) + K_t^0(a,f,c,g) \\ + K_{t+1}^s(f+1,T,g+1,T) \end{array} \right\}, \quad (3.39)$$

where $2 \leq t \leq T-1$ and $t \leq a, c \leq T$. If $t = 1$, we use (3.39) only with $a = c = 1$.

We begin the algorithm with $K_T^1(T,T) = K_T^2(T,T) = 0$ and work backwards for each of the retailers, followed by the warehouse and then the source node until we compute $K_1^s(1,T,1,T)$ which is the minimum cost attainable in the network.

Clearly, the computational effort involved in computing the minimum cost attainable in the network increases exponentially in the number of retailers. However, the properties of an extreme flow allow us to cut down the computational effort considerably.

Theorem 3.1. If the flow into a node $(0,t)$ is an extreme flow and is equal to $\sum_{i=a}^{b} r_i^1 + \sum_{i=c}^{d} r_i^2$, where $t \leq a \leq b \leq T$ and $t \leq c \leq d \leq T$, then $(a,b)$ intersects $(c,d)$ on the time axis, i.e. it is impossible for either $a \leq b < c \leq d$ or $c \leq d < a \leq b$ to hold.

Proof of Theorem 3.1.

Assume $a \leq b < c \leq d$. Clearly, the $\sum_{i=c}^{d} r_i^2$ units must be carried at the warehouse level until period $c$. This must be the case since to insure feasibility, $I_\tau^2 > 0$ for $t-1 \leq \tau \leq c-1$ and for any of the $\sum_{i=c}^{d} r_i^2$ units to be carried at the retailer level prior to period $c$ would require $x_\tau^2 > 0$ for some $t \leq \tau \leq c-1$. This is impossible by the property of extreme flows. Therefore, $I_\tau^0 > 0$ for $\tau$, where $t \leq \tau \leq c-1$.

By assumption $b < c$ and $r_{b+1}^1$ must be satisfied by period $b+1$ to maintain feasibility. Therefore, $x_\tau^0 > 0$ for $\tau$, where $t < \tau \leq b+1 \leq c$. Thus $x_{\tau'}^0 \cdot I_{\tau'-1}^0 > 0$ for some $\tau'$, where $t+1 \leq \tau' \leq c$ which contradicts the extreme flow property. Therefore, it cannot be the case that $a \leq b < c \leq d$. The proof for the case where $c \leq d < a \leq b$ simply reverses the role of the retailers in the above argument. □

It may be possible to improve the computational efficiency of the dynamic programming algorithm further through the use of bounds on the optimal solution value. Such bounds will be discussed in the following section.

## 3.6 Bounds on the Optimal Solution Value for One Warehouse, N Retailer Problems

The dynamic programming algorithm presented in Section 3.5 computes, for each node $(j,t)$, the optimal cost of shipping a group of exact requirements from that node to their ultimate destinations. If the cost of such a partially completed production inventory plan exceeds the cost of a known feasible solution to the entire problem, then we can omit consideration of any completions of this partial plan. This is clearly the case if all the cost functions are non-negative, which is a realistic assumption.

Obviously, the closer the known feasible solution value is to the optimum, the more likely it is that this bounding procedure will enable us to eliminate prospective solutions and thus save computational effort. We will discuss two methods of determining feasible solutions that will hopefully have this property. To simplify the discussion, we will again only explicitly consider the class where $N = 2$. The discussion is easily extended for larger values of N.

The first method consists of $N+1 = 3$ applications of the Wagner-Whitin single facility algorithm discussed in Section 3.3. Begin by solving for the best production inventory plan for the retailers, where each one is treated as an independent single facility problem. Let us

now assume that this portion of the one warehouse, 2 retailer plan is fixed. Consider the warehouse node $(0,t)$. Define $r_t^0 = \sum_{j=1}^{2} x_t^j$ which is determined by the choice of the production plans for the retailers. Therefore, given the production plans for the retailers, the problem reduces to a single facility problem with requirements $r_t^0$ and costs $H_t^0$ and $P_t^0$ which can again be solved by the Wagner-Whitin algorithm.

This plan guarantees feasibility by satisfying all the demands at the retailers and insuring that the inventory balance constraints at the warehouse nodes are met. It is also clear that this feasible solution will be an extreme flow. Furthermore, the value of any feasible solution is an upper bound on the optimal solution value. This method of decomposing the system is similar to the method proposed by Kalymon (1972) for general arborescent systems.

Since the optimization at the retailer level does not take the warehouse costs into account, the feasible solution obtained by this technique need not be the overall optimal solution. We can, for a particular cost structure at the warehouse, revise the production costs at the retailer to reflect, at least in part, the warehouse costs. Consider the node $(0,t)$ in the single source, network flow representation of the one warehouse, $N$ retailer problem (Figure 3.4).

If $(x_t^1+x_t^2) > 0$, then

$$I_{t-1}^0 + x_t^0 \geq x_t^1 + x_t^2$$

to maintain feasibility. By the property of extreme flows $I_{t-1}^0 \cdot x_t^0 = 0$ and so either

$$I_{t-1}^{0} \geq x_t^1 + x_t^2 \quad \text{or}$$

$$x_t^0 \geq x_t^1 + x_t^2.$$

Therefore, a cost of at least $\min(H_{t-1}^0(x_t^1+x_t^2);\ P_t^0(x_t^1+x_t^2))$ is incurred at the warehouse level by fixing $x_t^1 + x_t^2$ at the retailer level.

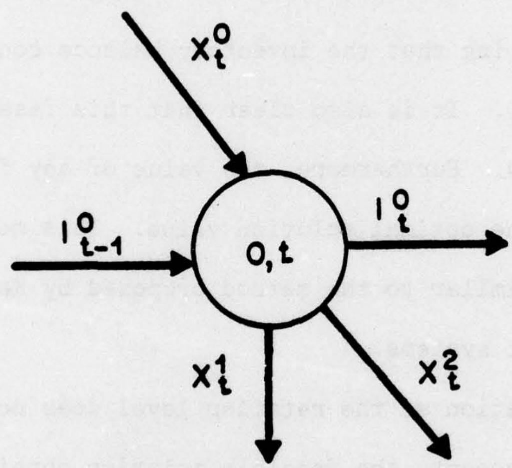


FIGURE 3.4 NODE (O, t) IN THE NETWORK FLOW REPRESENTATION

If

$$H_t^0(x) = \begin{cases} 0 & \text{if } x = 0, \\ H_t^0 + h_t^0 x & \text{if } x > 0, \end{cases} \quad \text{and}$$

$$P_t^0(x) = \begin{cases} 0 & \text{if } x = 0, \\ K_t^0 + p_t^0 x & \text{if } x > 0, \end{cases}$$

where $t = 1,\ldots,T$, then we can revise the production costs at the

retailers to reflect the cost that must be incurred at the warehouse level by fixing the production levels at the retailers. That is, set

$$\overline{K}_t^i(x) = \begin{cases} 0 & , \text{ if } x = 0 \\ K_t^i(x) + \min(\frac{H_t^0}{2}, \frac{K_t^0}{2}) + \min(h_t^0 x, p_t^0 x), & \text{ if } x > 0 \end{cases}, \text{ for } i = 1,2. \quad (3.40)$$

The $H_t^0/2$ and $K_t^0/2$ are present to insure that if both retailers produce in period $t$, only a single fixed charge of either $H_t^0$ or $K_t^0$ is incurred at the warehouse. (A similar revision in costs can be made if the warehouse costs have either a fixed charge at zero or a linear charge but not both.)

Thus we may summarize the technique to find a feasible solution as follows:

(1) For each retailer $i$, use the Wagner-Whitin algorithm with costs $\overline{K}_t^i$ and $H_t^i$ and requirements $r_t^i$ to compute a production plan for each retailer.

(2) Compute $r_t^0 = x_t^1 + x_t^2$ for all $t$.

(3) Solve for the best production plan at the warehouse using a Wagner-Whitin algorithm with costs $K_t^0$, $H_t^0$ and requirements $r_t^0$.

(4) Compute the overall cost of this production plan using the original costs for the network.

An alternative technique to find a feasible solution for this system is analogous to the separate retailing policy for the continuous

demand case. Simply split the one warehouse, N retailer system into N one-retailer systems and solve each of these systems independently using Zangwill's algorithm. This technique, however, requires much more computational effort than the method described above which used the Wagner-Whitin procedure.

We can also determine the quality of the feasible solution determined by either of the two heuristics described above by comparing the value of the solution to the value of a lower bound on the optimal solution. One possible lower bound can be computed by summing the costs of the Wagner-Whitin plans for each of the retailers, omitting the warehouse costs except in the cost $\overline{K}_t^i$. That is, these plans are determined <u>and</u> evaluated with the revised production cost $\overline{K}_t^i$ for retailer i. Clearly, no plan can be found with revised costs lower than this for the retailers and $\overline{K}_t^i$ was chosen to reflect the minimum cost added at the warehouse for any plan at the retailers.

An alternative lower bound may be determined by computing the Wagner-Whitin solution for the single facility with production costs $K_t^0$, holding costs $\overline{H}_t(x) = \min(H_t^0(x), H_t^1(x), H_t^2(x))$ and requirements $r_t^1 + r_t^2$. We can demonstrate that this is indeed a lower bound, if we examine the mathematical programming model for the one warehouse, 2 retailer system, i.e. (3.1)-(3.4) with N = 2. If we define $J_i = I_i^0 + I_i^1 + I_i^2$, we can rewrite the inventory balance equations, (3.2) and (3.3) as

$$x_t^0 + J_{t-1}^0 - J_t^0 = r_t^1 + r_t^2. \quad (3.41)$$

By the definition of $\overline{H}$, it is clear that

$$\overline{H}_t(I_t^0) \leq H_t^0(I_t^0).$$

Furthermore,

$$\overline{H}_t(I_t^0+I_t^1) \leq \overline{H}_t(I_t^0) + \overline{H}_t(I_t^1) \quad \text{by concavity}$$

$$\leq H_t^0(I_t^0) + H_t^1(I_t^1).$$

Similarly,

$$\overline{H}_t(J_i) \leq H_t^0(I_t^0) + H_t^1(I_t^1) + H_t^2(I_t^2).$$

Therefore, the objective function value for this relaxed mathematical program with constraint (3.41) and holding cost function $\overline{H}_t(J_i)$ is a lower bound on the optimal solution value for the original problem.

These lower bounds can be achieved when certain restrictions are placed on the cost structure of the system. These special cases will be discussed in the following section.

## 3.7 Exact Solutions for Restricted Cost Structures for One Warehouse, N Retailer Problems

It is also possible to use the Wagner-Whitin algorithm to get the optimal solutions for certain special cases of the cost structure in the one warehouse, $N$ retailer system.

Theorem 3.2. If $K_t^0(x) = px$, then the optimal solution for the one warehouse, $N$ retailer system may be found by solving $N$ independent single facility problems, one for each retailer.

Proof of Theorem 3.2.

If $K_t^0(x) = px$, we may ignore the per unit production costs altogether since the total number of units produced at the warehouse must equal $\sum_{t=1}^{T} \sum_{i=1}^{N} r_t^i$. Therefore, the cost of production at the warehouse may be assumed to be 0. Clearly, no units will be carried at the warehouse from period $t$ to $t+1$ provided $H_t^0$ is non-negative. Summing the costs for the $N$ independent Wagner-Whitin solutions is clearly a lower bound on the optimal cost. If we set $x_t^0 = \sum_{i=1}^{N} x_t^i$ for $t = 1,2,\ldots,T$, then the additional charge for this part of the production plan is 0 and the overall cost of the policy achieves the lower bound. □

Although the assumption that $K_t^0(x) = px$ may seem unrealistic, there are cases where such a cost structure is likely. If the warehouse receives deliveries each time period from its supplier, then there may be no additional fixed cost associated with placing orders in a particular policy. Similarly, if shipments are made each period from the warehouse to each of the retailers, then there may be no additional fixed set-up cost associated with a retailer placing an order in a particular policy. In this case, we may show

Theorem 3.3. If for all $t$ and $i$,

$$K_t^i(x) = p^i x, \text{ and}$$

$$H_t^0(x) \leq H_t^i(x),$$

then the optimal solution for the one warehouse, $N$ retailer system may

be found by solving the single facility problem with costs $K_t^0(x)$, $H_t^0(x)$ and requirements $r_t^0 = \sum_{i=1}^{N} r_t^i$.

Proof of Theorem 3.3.

As in the proof of Theorem 3.2 for the warehouse, we may ignore the production costs at the retailers entirely in this case. Now suppose that $I$ units are held from period $t$ to $t+1$ at retailer $i$. Clearly, we can decrease the preceding retailer production by $I$ units. If those $I$ units are carried at the warehouse until period $t+1$, a lower cost is incurred since $H_t^0(x) \leq H_t^i(x)$ and $H_t^0$ is concave. Adding a retailer $i$ production in period $t$ adds nothing to the cost. Therefore, all units are carried at the warehouse in an optimal policy.

This case is analogous to the second lower bound on the optimal solution value discussed in the preceding section. By that same argument, the optimal cost for the single facility problem with costs $K_t^0(x)$, $\bar{H}_t^0(x) = \min_i(H_t^i(x)) = H_t^0(x)$, and requirements $r_t^0 = \sum_{i=1}^{N} r_t^i$ is a lower bound on the overall optimal cost. Since the production costs at the warehouse are ostensibly zero, this lower bound is achieved by following the plan for the warehouse discussed above and simply setting $x_t^i = r_t^i$ for all $i$ and $t$. □

A similar result can be proven if retailer $i$ has lower holding cost over all $t$ than the warehouse has.

Corollary 3.4. If $K_t^0(x) = K + p^0 x$, $K_t^i(x) = p^i x$ for all $t$ and $i$ and for each $i$ either

(a) $H_t^0(x) \leq H_t^i(x)$, for all $t$ and $x \geq 0$

or (b) $H_t^i(x) = h^i \cdot x < h^0 \cdot x = H_t^0(x)$, for all $t$, and $x \geq 0$,

then the optimal solution of the one warehouse, $N$ retailer system may be found by solving an appropriate single facility problem.

Proof of Corollary 3.4.

As in Theorem 3.3, we may ignore the retailer production costs totally. If $H_t^0(x) \leq H_t^i(x)$, then no units are carried at retailer $i$ in an optimal solution. If for all $i$, this is the case then the corollary reduces to Theorem 3.3.

Suppose $H_t^i(x) = h^i \cdot x < h^0 \cdot x = H_t^0(x)$ for some $i$ and suppose $r_t^i$ is carried at the warehouse from some period $\tau$ to some period $\tau + \delta$. Clearly, we may reduce $I_\tau^0$ by $r_t^i$ units and add a retailer $i$ production in period $\tau$. This must reduce the holding costs over the periods $\tau$ to $\tau + \delta$ since $h^i \cdot x < h^0 \cdot x$. Therefore, no units are carried at the warehouse for retailer $i$.

Thus, the requirements for a particular retailer are either carried exclusively at the warehouse or exclusively at the retailer depending on whether case (a) or (b) holds concerning the holding costs. Denote the retailers that satisfy (a) by $1,2,\ldots,\alpha$.

It is also clear that if the warehouse produces in period $t$, then it produces for all the retailers. If not, then the holding costs may be reduced by decreasing the preceding warehouse production and increasing the production in period $t$. By assumption, this cannot increase the production costs at the warehouse.

If each time the warehouse produces it produces for all the retailers, then it is clear that the system has common regeneration points. Therefore, we may solve a single facility problem with production costs of $K_t^0$ and requirements of $\sum_{i=1}^{N} r_t^i$ in period $t$, provided that we charge

the holding costs properly. In particular, if the requirements for period $t+1$ are carried from period $t$ to $t+1$, the system holding costs are $H_t^0(\sum_{i=1}^{\alpha} r_t^i) + \sum_{i=\alpha+1}^{N} h^i \cdot r_t^i$. By suitable redefinition of (3.12) in the Wagner-Whitin recursion, we can solve this problem by that algorithm with some additional computation but no increase in the number of decisions. □

As was mentioned in Chapter 1, an extension of the Wagner-Whitin model that has received a great deal of attention is the problem with capacity constraints. Florian and Klein (1971), Lambrecht and Vander Eecken (1978) and Baker, Dixon, Magazine and Silver (1978) have all proposed algorithms to solve this particular problem.

Theorems 3.2, 3.3 and Corollary 3.4 demonstrate that under certain restrictions on the cost structure that one can solve the one warehouse, $N$ retailer problem by the solution of single facility problems using the Wagner-Whitin algorithm. Similarly, if there are production capacities on some of the arcs, then for certain cost structures it is possible to solve the one warehouse, $N$ retailer capacitated system by the solution of single facility, capacity constrained problems.

Corollary 3.5. If $K_t^0(x) = px$ and there are capacity constraints $x_t^i \leq c_t^i$, $i = 1,\ldots,N$, $t = 1,\ldots,T$, then the optimal solution for the one warehouse, $N$ retailer system may be found by solving the $N$ independent single facility, capacity constrained problems.

Proof of Corollary 3.5.

As in the proof of Theorem 3.2, we may ignore the production costs at the warehouse entirely, and also no units need by carried at the warehouse. The solution for the general one warehouse, $N$ retailer capacity

constrained problem is bounded below by the sum of the costs of the optimal solutions for the $N$ independent single facility capacity constrained problem. For this cost structure the bound is achieved. □

In a similar fashion, we may demonstrate

Corollary 3.6. If for all $t$ and $i > 0$ $K_t^i(x) = p^i x$, $H_t^0(x) \le H_t^i(x)$, and $x_t^0 \le c_t^0$, then the optimal solution for the one warehouse, $N$ retailer capacity constrained system may be found by solving the capacity constrained single facility problem with costs $K_t^0(x)$, $H_t^0(x)$ and requirements $r_t^0 = \sum_{i=1}^{N} r_t^i$. □

A similar restatement of Corollary 3.4 cannot be made because of the need to shift units produced at the warehouse to demonstrate common regeneration points. The capacity constraints on the production arcs at the warehouse preclude this argument.

## 3.8 Summary

In this chapter, we have examined the dynamic one warehouse, $N$ retailer production inventory system. This problem is a generalization of the 2-stage serial system discussed by Zangwill. The dynamic programming algorithm presented extends the concepts of Zangwill's algorithm to this more general problem. There are some computational simplifications that can be made and upper and lower bounds for the optimal solution value are discussed. The method for determining upper bounds is a heuristic technique to determine "good" feasible solutions.

Although the dynamic programming algorithm may seem intractable for problems with a large number of retailers, there are a number of

special cases where single facility algorithms may be applied to determine the optimal solution. Further investigation is necessary to establish when these much simpler computational techniques yield solutions that are either optimal or sufficiently close to optimal.

# CHAPTER 4

# CONCLUSION

In this thesis we have examined one warehouse, N retailer production inventory systems with both continuous and dynamic demand rates. For the continuous demand rate case, we have demonstrated that a single cycle policy need not be optimal for various classes of production policies. On the other hand, single cycle policies are optimal for certain classes of problems and the stationarity and renewal points present in such policies along with ease of computation make them especially appealing with regard to implementation.

Further research is indicated to determine bounds on the cost difference between the overall optimal solution and the best single cycle policy for those cases where it is not the optimal solution. In addition, direct comparisons of the iterative procedures presented in this thesis and the existing branch and bound algorithm (Graves and Schwarz, 1977) to determine single cycle policies should be performed. Particular attention should be paid to the relative effectiveness of these techniques for solving general arborescent problems through the examination of many one warehouse, N retailer problems.

Also, it was demonstrated in Chapter 2 that different one warehouse, N retailer problems with identical set-up and holding cost ratios have the same single cycle solution. An interesting sensitivity analysis question would be how much these ratios can be perturbed without changing this identical solution property. Finally, one warehouse, N retailer problems with less restrictive continuous demand assumptions also warrant

investigation. The deterministic single cycle solution may also serve as a good starting point in an algorithm to determine operating policies for two-echelon inventory systems with probabilistic demands (Muckstadt, 1977).

For the dynamic demand case, we have presented a dynamic programming algorithm to determine optimal policies along with computational simplifications that can be made. In addition, upper and lower bounds on the optimal solution value were discussed. Selected special cost structure cases where single facility techniques can be used to determine optimal policies were also introduced.

Further research and computational experience to determine the quality of the solutions obtained by single facility methods for general cost structure one warehouse, N retailer problems is indicated. Similarly, the one warehouse, N retailer problem with capacity constraints also warrants further attention. Dynamic one warehouse, N retailer problems with stochastic demands should also be investigated.

Finally, the techniques presented in this thesis should be applied to the other multi-echelon structures both with continuous and dynamic demands. It is possible that an iterative procedure similar to the one presented in this thesis can be used to determine the integer multiples of the basic lot size in the operating policies that have been proposed for serial and assembly systems. In addition, the relation of these policies to overall optimality without the restrictions of stationarity and integer multiples can be investigated in a manner similar to this thesis.

Finally, the techniques developed in this thesis should be applied to determine operating policies for actual one warehouse, N retailer

production inventory systems. Such systems can be found in many industrial and military situations and hopefully methods presented here can lead to more efficient and economical operations.

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In this thesis ~~we examine~~ the one warehouse, N retailer production inventory system is examined. We present properties of optimal operating policies and methods to determine various operating policies given a fixed cost for set-up and an inventory carrying charge at each facility. We assume that the external demands on this system occur either at a known continuous rate that is stationary over an infinite time horizon, or at a known rate that may vary in each of a finite number of periods. We will refer to the former case as a continuous one warehouse, N retailer problem and the latter as a dynamic demand problem.


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A one warehouse, N retailer system is a special case of the more general arborescent production inventory system. We begin by examining the previous research on systems with this arborescent structure as well as reviewing the literature dealing with the serial and assembly multi-echelon production inventory systems.

The continuous one warehouse, N retailer problem is then examined in detail. The basic model is introduced as well as some previously solved special cases. Several basic production policies that have been suggested for this system are reviewed and properties of an overall optimal solution are discussed. ~~We develop~~ optimal and heuristic algorithms are developed to determine the values of the parameters in "single cycle" policies (Schwarz, 1973) and make comparisons based on both the quality of solutions obtained and the computational effort. It has been conjectured by Graves and Schwarz (1977) that these single cycle policies are optimal for certain larger classes of production plans and we demonstrate that this is not the case. Finally, ~~we discuss~~ the class of multiple cycling policies and their relation to optimality, is discussed.

We then explore the dynamic one warehouse, N retailer problem. The basic model is introduced and both the Wagner-Whitin (1958) and the Zangwill (1966) algorithms are reviewed. Veinott (1969) has suggested the extension of Zangwill's algorithm to the case of N retailers, and we present this extension along with computational simplifications. Easily computer upper and lower bounds on the optimal solution value are developed. Exact solution techniques for several special cases for the cost structure are also discussed. Finally, areas for future investigation are suggested.